

# Quantum teleportation with close-to-maximal entanglement from a beam splitter

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We investigate theoretically quantum teleportation by means of the number-sum and phase-difference variables. We show that a unitary beam-splitter transformation turns two-mode Fock-states into close-to-maximally entangled states, in this case close approximations of the relative-phase eigenstates. These states could be created experimentally using on-demand single-photon sources, but also with any second-quantized bosonic system (e.g., trapped ions, Bose-Einstein condensates). We show that such “quasi-EPR” states can yield near-unity fidelity for the teleportation of coherent states and of “Schrödinger cat” states.

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## I. INTRODUCTION

Quantum teleportation [1,2], “the disembodied transport of an unknown quantum state from one place to another” [3], is an important component of quantum information. It holds great promise for communication between quantum computers, the realization of quantum gates [4] and the implementation of quantum error correction [5]. Quantum teleportation is based on maximally entangled states, a purely quantum mechanical feature, initially little noticed outside research on the fundamental issues of quantum theory such as the Einstein-Podolsky-Rosen (EPR) paradox [6]. Producing, preserving, and detecting high quality entanglement is an experimental challenge in making reliable quantum teleporters (and in quantum information in general). Initial experiments based on discrete and finite Hilbert spaces have been successful in proving the principle, but hindered by low efficiencies in the production and detection of entangled photons [7–9]. The use of continuous quantum variables for teleportation [2,10–13] offers more straightforward detection methods and also the interesting feature of an infinite Hilbert space, much richer in possibilities for encoding quantum information. The first continuous-variable teleportation experiment [3,10] used quadrature-squeezed electromagnetic fields and beam-splitter entanglement, and was based on the experimental realization of the EPR paradox [14] (see also [15,16]) using continuous quantum optical variables [17,18].

Another set of interesting, in part continuous, variables is constituted by the photon number and the phase, which are canonically conjugate in the same sense as energy and time [19]. The use of these variables has been proposed to realize the EPR paradox [17] and the corresponding maximally entangled states are therefore usable for teleportation by means of the detection of the photon number difference and phase sum [12], or of the photon number sum and phase difference [13,20]. The latter proposal, besides being more practical and connected to the actively explored field of Heisenberg-limited interferometry [21–25], has the advantage to be suited to experimental realization with bright squeezed sources, such as the ones demonstrated in Ref. [15,16]. In Ref. [13], Cochrane, Milburn, and Munro (CMM) study

quantum teleportation using phase-difference eigenstates. In this article, we adopt the definition of Luis and Sánchez-Soto [26] of phase-difference eigenstates, generalized to *relative-phase* eigenstates [27], and investigate the significance of such number-phase EPR states in the context of number-phase teleportation.

Because the creation of such states in the laboratory is still a challenge for more than two photons [27], we explore the use of approximate EPR states, or quasi-EPR states, such as created when a twin Fock state such as  $|n\rangle_a|n\rangle_b$  passes through a lossless beam splitter—which would be very simply realizable with on-demand single-photon sources. CMM found that, for such states, the teleportation fidelity is bounded by the classical limit of 50% for an arbitrary coherent state, but reaches 100% for a “Schrödinger cat” state proportional to,  $|\alpha\rangle+|-\alpha\rangle$  [13]. This is due, as CMM points out, to the fact that half the quantum amplitudes of this entangled state are null, and does not affect the cat state which presents the same characteristic (provided that the nonzero amplitudes coincide).

The gist of this paper is to show that this nulling of half of the EPR amplitudes corresponds to a very narrow set of experimental conditions and disappears as soon as the beam splitter or its input Fock state are not perfectly balanced (“perfectly” referring, for the beam splitter, to the Heisenberg limit). As a result, we show that a wider set of quasi-EPR states than considered by CMM do yield near-unity fidelity for teleportation of both coherent and cat states.

In Sec. II we derive and evaluate close-to-maximally entangled, or quasi-EPR, states that can be created by sending two-mode Fock states through a lossless beam splitter. In Sec. III we describe number-phase teleportation for ideal EPR states, and for quasi-EPR states. We then analyze the fidelity of quasi-EPR based teleportation of coherent and cat states.

## II. GENERATION OF CLOSE-TO-MAXIMALLY ENTANGLED STATES BY A BEAM SPLITTER

### A. The Schwinger representation

We begin by recalling the definition of the Schwinger representation [28], widely used in quantum optics, of a non-degenerate two-mode field in terms of a fictitious spin. This spin is defined as

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$$\mathbf{J} = \begin{pmatrix} J_x \\ J_y \\ J_z \end{pmatrix} = \frac{1}{2} \begin{pmatrix} a^\dagger b + b^\dagger a \\ -i(a^\dagger b - b^\dagger a) \\ a^\dagger a - b^\dagger b \end{pmatrix}, \quad (1)$$

$$\mathbf{J}^2 = \frac{a^\dagger a + b^\dagger b}{2} \left( \frac{a^\dagger a + b^\dagger b}{2} + 1 \right), \quad (2)$$

where  $a$  and  $b$  are the photon annihilation operators of each mode. The physical meaning of these operators is the following [26,25]:  $\mathbf{J}^2$  represents the total photon number,  $J_z$  the photon number difference between the two modes, and  $J_{x,y}$  are the phase difference, or interference, quadratures. It stems from this that  $e^{i\theta J_z}$  is the relative phase shift operator and  $e^{i\theta J_x}$  is the rotation carried out by a beam splitter (homo-/heterodyne measurements). The eigenstates of the fictitious spin are the two-mode Fock states

$$|j m\rangle_z = |n_a\rangle_a |n_b\rangle_b, \quad (3)$$

and the respective eigenvalues of  $J^2$  and  $J_z$  are given by

$$j = \frac{n_a + n_b}{2} = \frac{N}{2}, \quad (4)$$

$$m = \frac{n_a - n_b}{2}. \quad (5)$$

The Schwinger representation thus makes use of the homomorphism from  $SU(2)$  onto  $SO(3)$ , which allows one to represent any unitary operation on the two-mode field by a rotation. The general  $SU(2)$  transformation

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} \cos \frac{\beta}{2} e^{i/2(\alpha+\gamma)} & \sin \frac{\beta}{2} e^{i/2(\alpha-\gamma)} \\ -\sin \frac{\beta}{2} e^{-i/2(\alpha-\gamma)} & \cos \frac{\beta}{2} e^{-i/2(\alpha+\gamma)} \end{pmatrix} \begin{pmatrix} a_0 \\ b_0 \end{pmatrix}, \quad (6)$$

where  $\alpha$ ,  $\beta$ , and  $\gamma$  are the Euler angles, corresponds to the rotation operator  $e^{-i\alpha J_z} e^{-i\beta J_y} e^{-i\gamma J_z}$  ( $\hbar=1$ ). A lossless beam splitter corresponds to the values

$$\alpha = -\gamma = \pi/2, \quad \beta = \pm 2 \arccos \sqrt{R}, \quad (7)$$

where  $R$  is the reflectivity of the beam splitter (the transmittivity  $T$  is such that  $R+T=1$ ).

### B. Ideal number-phase EPR states

By definition, a maximally entangled two-mode state, or EPR state, is a two-mode quantum state

$$|\text{EPR}\rangle = \sum_{k,l} s_{kl} |k\rangle_a |l\rangle_b, \quad (8)$$

such that any reduced (single-mode) density matrix of this state  $\text{Tr}_{a,b}(|\text{EPR}\rangle\langle\text{EPR}|)$  is proportional to the identity matrix, which yields

$$\sum_k s_{kl} s_{k'l'}^* = \delta_{ll'}, \quad \sum_l s_{kl} s_{k'l}^* = \delta_{kk'}. \quad (9)$$

An example of the EPR state is the eigenstate, introduced by Luis and Sánchez-Soto [26], of the operators of the photon-number sum and phase difference of two modes  $a$  and  $b$ . Heeding the point made by Trifonov *et al.* [27], we will call this state a *relative-phase* eigenstate rather than a phase-difference eigenstate, thus recalling that the formal definition of a two-mode quantum phase difference operator does not coincide with the (problematic) definition of two single-mode quantum phase operators [29]. The relative-phase eigenstate is

$$|\phi_r^{(N)}\rangle = \frac{1}{\sqrt{N+1}} \sum_{n=0}^N e^{in\phi_r^{(N)}} |n\rangle_a |N-n\rangle_b, \quad (10)$$

where  $\phi_r^{(N)} = \phi_0 + 2\pi r/(N+1)$  is the phase difference,  $N$  is the total photon number,  $\phi_0$  an arbitrary phase origin, and  $r \in [0, N]$ . The phase difference is adequately defined with resolution  $1/N$ , i.e. at the Heisenberg limit. In the Schwinger representation, Eq. (10) becomes

$$|\phi_r^{(2j)}\rangle = \frac{1}{\sqrt{2j+1}} \sum_{m=-j}^j e^{im\phi_r^{(2j)}} |jm\rangle_z. \quad (11)$$

Trifonov *et al.* introduced the more general state

$$|\{\theta^{(N)}\}\rangle = \frac{1}{\sqrt{N+1}} \sum_{n=0}^N e^{i\theta_n^{(N)}} |n\rangle_a |N-n\rangle_b, \quad (12)$$

which they stated can still be considered a relative-phase eigenstate, however involved or even arbitrary the real set  $\{\theta_n^{(N)}\}_n$  may be with respect to  $n$  [27]. Indeed, the whole basis can always be constructed by applying the  $N+1$  phase shift operators  $e^{i\phi_r^{(N)} J_z}$  to  $|\{\theta^{(N)}\}\rangle$ , with  $\{\theta_n^{(N)}\}_n$  being an initial relative-phase distribution between the two modes  $a$  and  $b$ . Nonetheless, we will show in Sec. III that successful quantum teleportation demands full initial knowledge of  $\{\theta_n^{(N)}\}_n$ , which is also the relative phase of the entanglement between Alice and Bob, and the number-phase teleportation protocol becomes very complicated if  $\theta_n^{(N)}$  is not linear in  $n$ .

Finally, we recall that maximal entanglement is only attained when  $N \rightarrow \infty$ .

### C. Generation of EPR and quasi-EPR states

The experimental realization of relative-phase eigenstates is an arduous problem. Recently, Trifonov *et al.* reported the experimental realization of a relative-phase eigenstate (12) for  $N=2$  [27]. Their astute method uses a nonbalanced beam splitter to create a two-mode state, all of whose amplitudes have equal modulus. This method is not general in the sense that it cannot work perfectly for  $N>2$ , as we will see later (and is immediate from Fig. 2). However, the use of a beam splitter to generate EPR or quasi-EPR states stems from quite general arguments indeed.

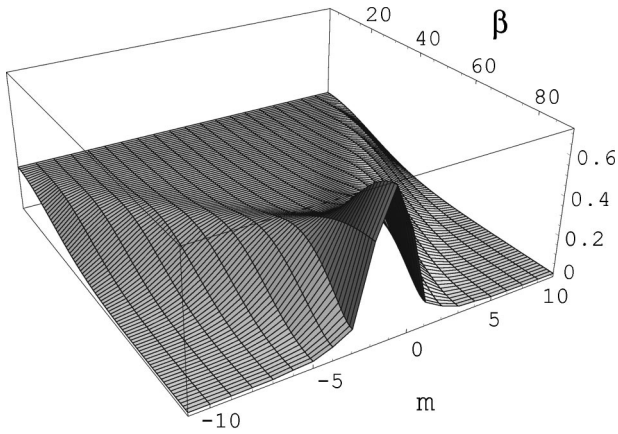


FIG. 1. Modulus of the quantum amplitudes of beam-splitter output  $|\phi_0^{(20)}(\beta)\rangle$  for a relative-phase eigenstate input versus the beam-splitter angle  $\beta$ .  $N=20$  photons.

The studies of Heisenberg-limited interferometry [21–25] have led to a thorough understanding of the subtle quantum optical properties of the beam splitter [30,22–25]. In particular, a balanced beam splitter swaps the amplitudes and phase properties of the impinging two-mode wave [31], and can also be used to entangle nonclassical optical fields [3,32,33]. As is readily seen in the Schwinger representation, a balanced beam splitter corresponds to a  $\pi/2$  rotation around  $X$  and therefore transforms a state from axis  $Z$  (intensity difference) to axis  $Y$  (phase difference).

In the case of an EPR state such as Eq. (11), the phase difference is squeezed and the intensity difference is anti-squeezed. Experimentally, this is achievable by sending an intensity-difference-squeezed state through a beam splitter [15]. To illustrate this, let us examine the beam-splitter output of the relative phase state (11):

$$|\phi_0^{(2j)}(\beta)\rangle = e^{-i\beta J_x} |\phi_0^{(2j)}\rangle \quad (13)$$

$$= \frac{1}{\sqrt{2j+1}} \sum_{m=-j}^j i^{-m} f_m^j(\beta) |jm\rangle_z, \quad (14)$$

where  $f_m^j(\beta) = \sum_{m'=-j}^j e^{im'(\phi_0^{(2j)} + \pi/2)} d_{mm'}^j(\beta)$ , with  $d_{mm'}^j(\beta) = \langle jm|e^{-i\beta J_y}|jm'\rangle$  a rotation matrix element taken real by convention and proportional to a Jacobi polynomial [34]. This state is displayed for  $\phi_0=0$  in Fig. 1. As expected, the result is a narrow state in photon number (i.e.,  $f_m^j \rightarrow 0$  very fast as  $|m| \rightarrow j$ ).

It is straightforward to see that sending  $|\phi_0^{(2j)}(\pi/2)\rangle$  through another, balanced ( $\beta = \pm\pi/2$ ), beam splitter will reconstruct the initial relative phase eigenstate  $|\phi_0^{(2j)}\rangle$ . (One can see that  $\beta \rightarrow \pi/2$  is necessary to maximal entanglement, since the spread of the state must cover all values of the projections of the spin.) Now, since  $|\phi_0^{(2j)}(\pi/2)\rangle$  contains but a few nonzero amplitudes, a reasonable method for generating close approximations of EPR states with a balanced beam splitter is to consider “quantum filtered” input states, which are derived from  $|\phi_0^{(2j)}(\pi/2)\rangle$  by keeping only the

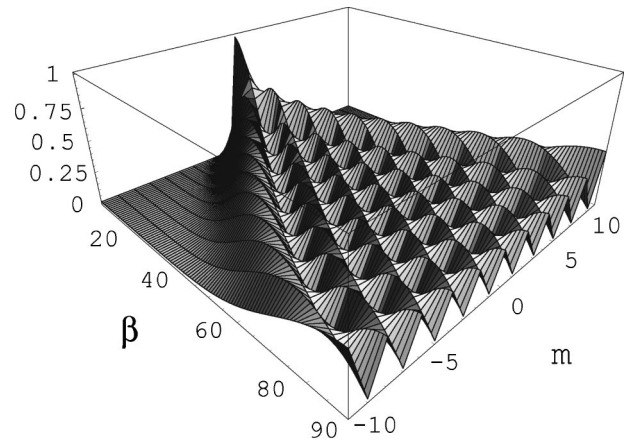


FIG. 2. Modulus of the quantum amplitudes of the beam-splitter output  $|\Psi_j^{(1)}(\beta)\rangle = |\Psi_{10}^{(1)}(\beta)\rangle$  versus the beam-splitter angle  $\beta$ .  $N=20$  photons.

lowest values of  $m$  (this could be viewed as a quantum Abbe-Porter experiment, with a low- $m$ -pass filter). These states are, for  $N=2j$  even,

$$|\Psi_j^{(1)}\rangle = |j0\rangle_z, \quad (15)$$

$$|\Psi_j^{(3)}\rangle = [f_0^j |j0\rangle_z + f_1^j |j1\rangle_z + f_{-1}^j |j-1\rangle_z] / C_1, \quad (16)$$

and so on, with  $C_\mu = (\sum_{m=-\mu}^\mu |f_m^j|^2)^{1/2}$ , and are

$$|\Psi_j^{(2)}\rangle = [|j\frac{1}{2}\rangle_z + |j-\frac{1}{2}\rangle_z] / \sqrt{2}, \quad (17)$$

$$|\Psi_j^{(4)}\rangle = [f_{\frac{1}{2}}^j |j\frac{1}{2}\rangle_z + f_{-\frac{1}{2}}^j |j-\frac{1}{2}\rangle_z + f_{\frac{3}{2}}^j |j\frac{3}{2}\rangle_z + f_{-\frac{3}{2}}^j |j-\frac{3}{2}\rangle_z] / C_{\frac{3}{2}}, \quad (18)$$

and so on, for  $N=2j$  odd. As we will now see, sending these states through a beam splitter gives output states closely resembling EPR states. We call these output states quasi-EPR states.

We start with the simplest one (15), which has already been considered by CMM. We denote the general state rotated by a beam splitter by

$$|jm'\rangle(\beta) = e^{-i\beta J_x} |jm'\rangle_z \quad (19)$$

$$= \sum_{m=-j}^j i^{m'-m} d_{mm'}^j(\beta) |jm\rangle_z. \quad (20)$$

Figure 2 displays the modulus of the quantum amplitudes of  $|\Psi_j^{(1)}(\beta)\rangle$  versus  $\beta$  [see Eq. (7)] and  $m$ .

One can see that  $\beta \rightarrow \pi/2$  is still necessary for maximal entanglement. The very value  $\beta = \pi/2$  leads to a problem, however, because every other amplitude of the state is zero, as is well known [30,25,13]. This was recalled by CMM when they investigated the use of this state as a teleportation channel and found that, because of this, teleportation fidelity was bounded by 50% (to the notable exception of Schrödinger cat states). This situation, however, is changed if one considers an ever-so-slightly imbalanced beam splitter: Fig.

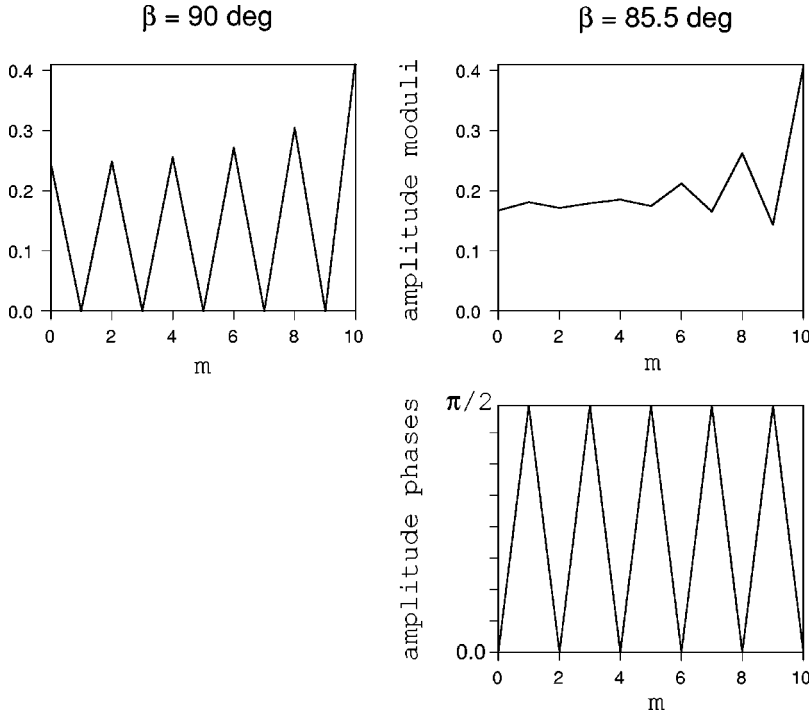


FIG. 3. Modulus and phase of the quantum amplitudes of  $|\Psi_{10}^{(1)}(90^\circ)\rangle$  (left) and  $|\Psi_{10}^{(1)}(85.5^\circ)\rangle$  (right). All nonzero amplitudes of  $|\Psi_{10}^{(1)}(90^\circ)\rangle$  have a phase equal to  $\pi/2$ .

3 displays the modulus and phase of the coefficients of  $|10\ 0(90^\circ)\rangle = |\Psi_{10}^{(1)}(90^\circ)\rangle$  and  $|10\ 0(85.5^\circ)\rangle = |\Psi_{10}^{(1)}(85.5^\circ)\rangle$ .

Clearly,  $|\Psi_{10}^{(1)}(85.5^\circ)\rangle$  is closer to an EPR state than  $|\Psi_{10}^{(1)}(90^\circ)\rangle$ : it has the same spread but much more even amplitudes, practically constant for  $-5 \leq m \leq 5$ , and no zeroes at all. The phase distribution is not constant and not simple but this just means that it is a general relative phase state of the form of Eq. (12), which is still a legitimate EPR state in the context we chose for the definition of the relative phase. In fact,  $|\Psi_{10}^{(1)}(85.5^\circ)\rangle$  is the best quasi-EPR state  $|\Psi_{10}^{(1)}(\beta)\rangle$ ,  $\forall \beta$ . In general, we find that the angle  $\beta_Q$  that gives the best quasi-EPR state is given by the following phenomenological formula:

$$\beta_Q = \frac{\pi}{2} \left( 1 - \frac{1}{N} \right), \quad (21)$$

which we have tested, to the best of our numerical capabilities Eq. (21), for  $N = 200, 2000$ , and  $20000$  photons: the resulting states  $|\Psi_j^{(1)}(\beta_Q)\rangle$  for  $j = 100, 1000$ , and  $10000$ , are plotted in Fig. 4. Note that the size of the low- $m$  flat region does scale proportionally to  $N$ .

Note that all digits are significant in the beam splitter's parameters of Fig. 4, which points to an interesting situation. Let us assume that on-demand single-photon sources become a reality (not an unreasonable assumption), which would allow the production of  $|\Psi_j^{(1)}\rangle_z$  in the laboratory. Equation (21) nevertheless poses a serious experimental constraint on the tolerance of the beam-splitter reflectivity  $R$ , because the required precision on  $\beta$ , i.e., on  $R$ , increases with  $N$  if one wants to resolve  $|\Psi_j^{(1)}(\beta_Q)\rangle$  from  $|\Psi_j^{(1)}(\pi/2)\rangle$  and its  $j$  inconvenient zeroes. Roughly,  $\Delta R \sim \Delta \beta \sim 1/N$ . Since a beam splitter using state-of-the-art optical coatings and polishing

cannot give more than  $\Delta R \sim 10^{-6}$ ,  $N$  cannot exceed  $10^6$  photons in this case. This ‘‘Heisenberg-limit’’-type sensitivity may be quite general, as hints of the same scaling have also been found in the tolerance to error of entanglement operations (optical pulse times) of  $N$  trapped ions [35]. In fact, by taking a closer look at Fig. 2, one can see that the amplitudes present  $1/N$ -period oscillations versus  $\beta$ . These oscillations are of significant contrast, with the state amplitudes often reaching zero. This, therefore, can pose a problem for experimentally defining  $|\Psi_j^{(1)}(\beta)\rangle$  as  $j$  increases.

This problem disappears, however, as soon as one uses a more elaborate input state, such as  $|\Psi_j^{(2)}\rangle$  (17). Such an input state could be obtained using stimulated emission from a single atom, starting from a  $|\Psi_j^{(1)}\rangle_z$  state and having the two beams shine simultaneously and noncollinearly on the excited atom. One will also want to have fast nonradiative decay from the ground state of the transition so as to prevent subsequent absorption, as in laser media. Figure 5 displays the state  $|\Psi_j^{(2)}(\beta)\rangle$ .

Even though the  $1/N$  oscillations are still present, they are significantly attenuated as there are no zero amplitudes, even at  $\beta = \pi/2$ . One can therefore use  $\beta = \pi/2$  in this case, which presents the non-negligible advantage of a *flat phase* distribution ( $3\pi/4, \forall m$ ) for this state  $|\Psi_j^{(2)}(\pi/2)\rangle$  [unlike  $|\Psi_j^{(1)}(\beta_Q)\rangle$  in Fig. 3]. This is of great importance for teleportation, as we will see in the next section. Finally, it is straightforward and unsurprising to show that the more elaborate (less low- $m$ -pass filtered) reconstructions  $|\Psi_j^{(3)}\rangle$  (16) and  $|\Psi_j^{(4)}\rangle$  (18) give even better results: more even amplitudes at still constant phase. We will, however, restrain our investigations to the two states  $|\Psi_j^{(1)}(\beta_Q)\rangle$  and  $|\Psi_j^{(2)}(\pi/2)\rangle$ , which are the simplest ones and are also within reasonable reach of foreseeable future technology.



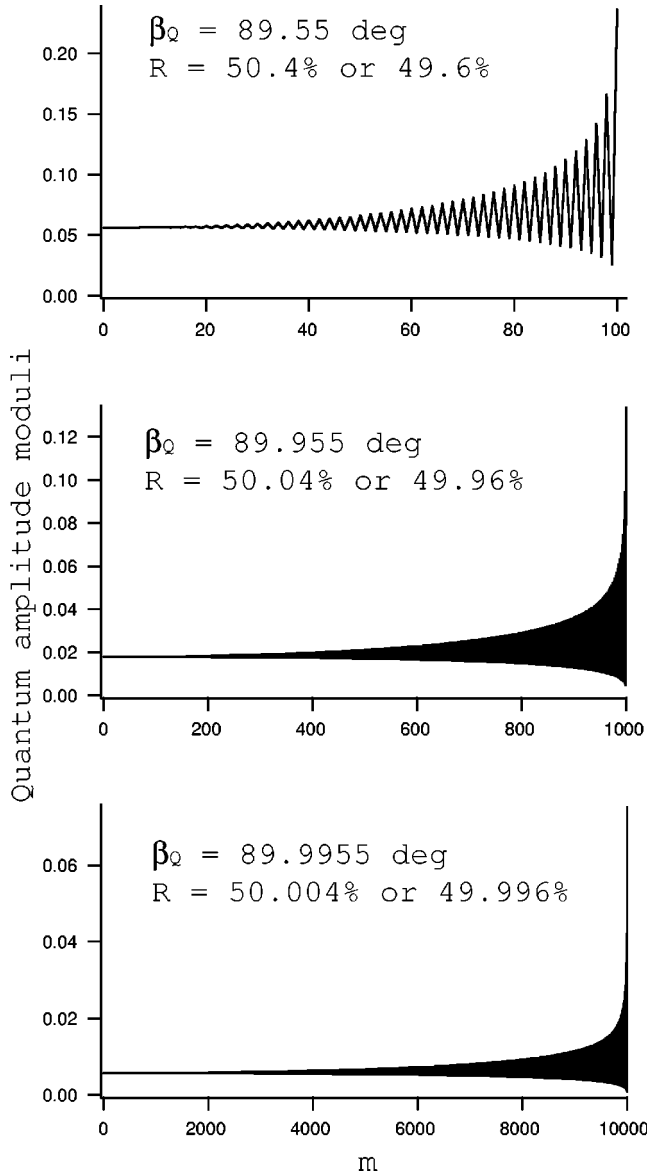


FIG. 4. Modulus of the quantum amplitudes of  $|\Psi_j^{(1)}(\beta_Q)\rangle$  for  $N=200, 2000,$  and  $20000$  photons (see Fig. 3 for  $N=20$ ). All states have an almost identical appearance, and are practically indistinguishable from perfect relative-phase eigenstates for  $-N/5 \leq m \leq N/5$ , approximately.

### III. NUMBER-PHASE QUANTUM TELEPORTATION

In this section we briefly recall the definition of number-phase teleportation [13] and extend it to general relative-phase states. We then consider the use of the quasi-EPR states that were derived above.

#### A. Ideal entanglement resource

Quantum teleportation relies on a maximally entangled state shared by the sender, Alice, and the receiver, Bob. The entanglement concerns two physical systems,  $\mathcal{A}$  and  $\mathcal{B}$ , respectively. Alice is in possession of  $\mathcal{A}$  and also of system  $\mathcal{T}$ , whose “target” state is the quantum information she needs to transmit. The teleportation process consists, for Alice, in

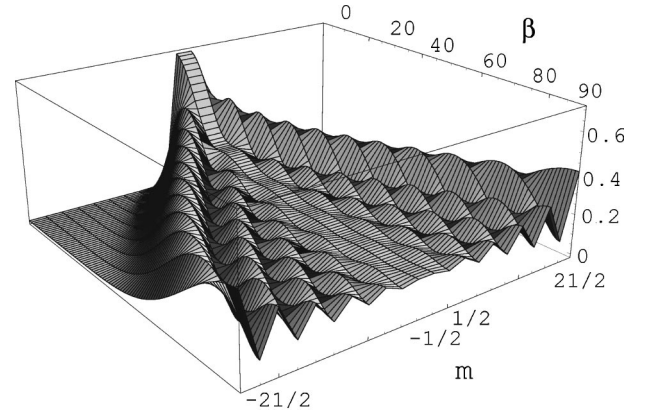


FIG. 5. Modulus of the quantum amplitudes of beam-splitter output  $|\Psi_{21/2}^{(2)}(\beta)\rangle$  versus the beam-splitter angle  $\beta$ .  $N=21$  photons.

making a joint measurement on  $\mathcal{T}$  and  $\mathcal{A}$  such that both are projected onto a maximally entangled state. This prevents Alice from obtaining any quantum information about the target state, which as such is destroyed in the process. In turn, said target state is transferred, by “entanglement transitivity” from  $\mathcal{T}$  to  $\mathcal{B}$ , i.e., to Bob, who may then reconstruct the exact target state on  $\mathcal{B}$  using the classically transmitted results of Alice’s measurements (which contain no quantum information whatsoever). The conceptually difficult part is to figure out what measurements should be used by Alice to maximally entangle her two systems  $\mathcal{A}$  and  $\mathcal{T}$ . This question was answered by Vaidman in connection with the EPR paradox [2].

In the case of number-phase teleportation, use is made of the commuting operators number sum  $\hat{N}_T + \hat{N}_A$  and (Hermitian) relative phase [26]

$$\hat{\phi}_{TA} = \sum_{N=0}^{\infty} \sum_{r=0}^{N+1} |\phi_r^{(N)}\rangle \phi_r^{(N)} \langle \phi_r^{(N)}|, \quad (22)$$

whose measurements project the joint  $\mathcal{T}$ - $\mathcal{A}$  state onto a joint eigenstate of the total number and the relative phase such as Eq. (10). If the same type of entangled state is shared between Alice and Bob, perfect teleportation can in principle be achieved. Let us consider the general case where the initial total state is

$$|\psi\rangle_{\mathcal{T}} \otimes |\phi_r^{(N)}\rangle_{AB} \equiv \sum_{m=0}^{\infty} c_m |m\rangle_{\mathcal{T}} \sum_{n=0}^N \frac{e^{in\phi_r^{(N)}}}{\sqrt{N+1}} |n\rangle_A |N-n\rangle_B. \quad (23)$$

We assume Alice’s measurements of  $\hat{N}_T + \hat{N}_A$  and of  $\hat{\phi}_{TA}$  yield the respective eigenvalues  $q$  and  $\phi_s^{(q)}$ . The phase difference measurement should be thought of as a Heisenberg-limited interferometric measurement, whose experimental implementation was proposed and numerically modeled, for the states considered here, in Refs. [23,25]. The joint TA state is thus left in  $|\phi_s^{(q)}\rangle_{TA}$ , and the total state after Alice’s measurement is

$$|\psi_M\rangle = e^{iq\phi_r^{(N)}} |\phi_s^{(q)}\rangle_{TA} \otimes |\psi_B\rangle_B, \quad (24)$$

$$|\psi_B\rangle_B = C(q) \sum_{k_0}^q e^{-ik(\phi_r^{(N)} + \phi_s^{(q)})} c_k |k+N-q\rangle_B, \quad (25)$$

where  $k_0 = \max[0, q-N]$  and  $C(q) = [\sum_{k=k_0}^q |c_k|^2]^{-1/2}$ . To exactly recover the target state, Bob must then perform, on  $B$ , a photon number shift [13] of  $q-N$  and a phase shift of  $\phi_r^{(N)} + \phi_s^{(q)}$ , i.e.,

$$|\psi_{out}\rangle_B = e^{i(\phi_r^{(N)} + \phi_s^{(q)})\hat{N}_B} \mathcal{P}_{q-N} |\psi_B\rangle_B. \quad (26)$$

(In the particular case where Alice's measurement results are  $q=N$  and  $\phi_s^{(q)} = -\phi_r^{(N)}$ , Bob does not have to do anything.) One can see from this that the phase distribution of the initial entanglement resource has to be corrected for, along with Alice's measurement result. *If this distribution is unknown or too complicated to correct, teleportation will fail.* This correction can also be made by Alice, by simply shifting her phase operator, i.e., using

$$e^{-i\phi_r^{(N)}J_z} \hat{\phi}_{TA} e^{i\phi_r^{(N)}J_z} \quad (27)$$

instead of  $\hat{\phi}_{TA}$ . [ $J_z = (\hat{N}_T - \hat{N}_A)/2$  here.]

Note that this requirement that the entanglement phase be perfectly known is a very general one and not at all specific to our particular choice of the optical phase variable for the teleportation protocol. This was pointed out by van Enk in Ref. [36].

In light of what we have already discussed, it is clear that a state such as  $|\Psi_j^{(2)}(\pi/2)\rangle$ , which has a flat phase distribution  $e^{in\phi_r^{(N)}} = e^{i3\pi/4}$ , is perfectly suited for this teleportation protocol. It is, however, interesting to investigate the use of the more exotic state  $|\Psi_j^{(1)}(\beta_Q)\rangle$  in this case. The phase distribution is neither flat nor simple (Fig. 3), one phenomenological description of it being  $e^{in\phi_r^{(N)}} = e^{in^2\pi/2}$ . We are thus in the case of a general relative phase state  $|\{\theta^{(N)}\}\rangle$  given by Eq. (12). If the initial entanglement is given by Eq. (12),

$$|\psi\rangle_T \otimes |\{\theta^{(N)}\}\rangle_{AB}, \quad (28)$$

then the phase difference operator is not given by Eq. (22) but by [27]

$$\hat{\phi}_{TA} = \sum_{N=0}^{\infty} \sum_{r=0}^{N+1} e^{i\phi_r^{(N)}J_z} |\{\theta^{(N)}\}\rangle \phi_r^{(N)} \langle\{\theta^{(N)}\}| e^{-i\phi_r^{(N)}J_z}, \quad (29)$$

and the postmeasurement total state is

$$|\psi_M\rangle = e^{i(q/2)\phi_s^{(q)}} |\phi_s^{(q)}\rangle_{TA} \otimes |\psi_B\rangle_B, \quad (30)$$

$$|\psi_B\rangle_B = C(q) \sum_{k=k_0}^q e^{i\theta_{q-k}^{(N)} - \theta_k^{(N)}} e^{-ik\phi_s^{(q)}} c_k |k+N-q\rangle_B, \quad (31)$$

which shows that Bob needs more than a mere phase shift to properly reconstruct  $|\psi\rangle$ : with  $\{\theta_n^{(N)}\}_n$  fully known, he needs the unitary transformation  $U_{\{\theta^{(N)}\}}$  that transforms  $|\{\theta^{(N)}\}\rangle$

into a ‘‘flat-phase’’ state for which  $\theta_n^{(N)} = \phi_0^{(N)} = \text{cst}$ ,  $\forall n$ . This transformation may be single-mode and applied by Bob, as  $U_{\{\theta^{(N)}\}}^B |\psi_B\rangle_B = |\psi_{out}\rangle_B$ , or two-mode and applied by Alice, by measuring  $U_{\{\theta^{(N)}\}}^A \hat{\phi}_{TA} U_{\{\theta^{(N)}\}}^{A\dagger}$ . In the aforementioned case of  $|\Psi_j^{(1)}(\beta_Q)\rangle$ , one possibility is

$$U_{\{\theta\}}^B = e^{i(-1)^q(\pi/2)\hat{N}_B^2}, \quad (32)$$

which can be interpreted as an intensity-dependent phase shift and might therefore be realizable with a Kerr nonlinearity.

It is somewhat puzzling that this additional step is needed if  $|\{\theta^{(N)}\}\rangle$  may indeed be considered as a legitimate relative phase eigenstate, since it is used in the corresponding relative phase measurement of Eq. (29). Our conclusion is that the definition (29) of the general relative-phase operator is not valid in the context of quantum teleportation, possibly because the full EPR nature of the eigenstate of Eq. (12) is used. Indeed, if one only cares about measuring the phase difference between the two modes, any type of initial relative phase distribution  $\{\theta_n^{(N)}\}_n$  will give the same result, as illustrated in Ref. [27]. This obviously ceases to be true when applying in a teleportation operation, where the complete nature of the entangled state must be known. In other words yet, even though the whole relative-phase eigenbasis — and hence the operator — may be generally defined based upon any general state  $|\{\theta^{(N)}\}\rangle$  with arbitrary phase distribution  $\{\theta_n^{(N)}\}_n$ , it does in fact matter for teleportation that  $U_{\{\theta^{(N)}\}}$  corresponds to a feasible physical measurement, thereby limiting the generality of relative phase states usable for teleportation.

## B. Quasi-EPR resource

We now turn to the use of quasi-EPR states as the entanglement resource, and show how arbitrary coherent and Schrödinger cat states can be successfully teleported. Our evaluation of teleportation performance will contain no assessment of decoherence and will be based on the pure-state fidelity

$$F = |\langle\psi_{out}|\psi\rangle|^2. \quad (33)$$

The entanglement resource is now a quasi-EPR state,

$$|\text{QEPR}\rangle_{AB} = \sum_{n=0}^N s_n^N |n\rangle_A |N-n\rangle_B, \quad (34)$$

where  $\sum_{n=0}^N |s_n^N|^2 = 1$ . As announced before, we only treat the cases of  $|\Psi_j^{(1)}(\beta_Q)\rangle$  and  $|\Psi_j^{(2)}(\pi/2)\rangle$ . Their respective decompositions in terms of Eq. (34) are found using Eqs. (3)–(5) and (19), and their amplitudes are, respectively,

$$s_n^N = i^{-n+(N/2)} d_{n-(N/2), 0}^{N/2}(\beta_Q), \quad (35)$$

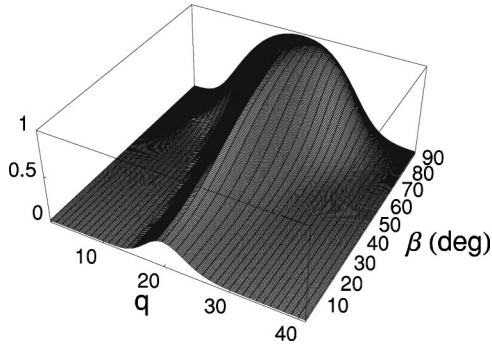


FIG. 6. Teleportation fidelity of an  $\alpha=3$  coherent state, using the quasi-EPR resource  $|\Psi_{21/2}^{(2)}(\beta)\rangle$  versus the beam-splitter angle  $\beta$  and the number-sum measurement result  $q$ .  $N=21$  photons.

$$s_n^N = \frac{1}{\sqrt{2}} i^{-n+(N/2)} \left[ d_{n-(N/2), (1/2)}^{N/2} \left( \frac{\pi}{2} \right) - i d_{n-(N/2), -1/2}^{N/2} \left( \frac{\pi}{2} \right) \right]. \quad (36)$$

The postmeasurement total state is

$$|\psi_M\rangle = |\phi_s^{(q)}\rangle_{TA} \otimes |\psi_B\rangle_B, \quad (37)$$

$$|\psi_B\rangle_B = C(q) \sum_{k=k_0}^q e^{-ik\phi_s^{(q)}} c_k s_{q-k}^N |k+N-q\rangle_B, \quad (38)$$

where  $C(q) = (\sum_{k=k_0}^q |c_k|^2 |s_{q-k}^N|^2)^{-1/2}$ . This yields

$$|\psi_{out}\rangle_B = C(q) \sum_{k=k_0}^q c_k s_{q-k}^N |k\rangle_B, \quad (39)$$

and the fidelity is

$$F(q) = \frac{\left| \sum_{k=k_0}^q |c_k|^2 s_{q-k}^N \right|^2}{\sum_{k=k_0}^q |c_k|^2 |s_{q-k}^N|^2}. \quad (40)$$

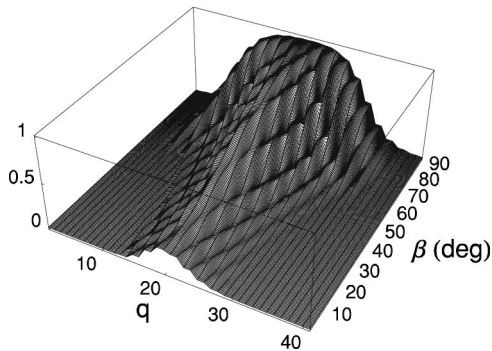


FIG. 7. Teleportation fidelity of an  $\alpha=3$  cat state, using the quasi-EPR resource  $|\Psi_{21/2}^{(2)}(\beta)\rangle$  versus the beam-splitter angle  $\beta$  and the number-sum measurement result  $q$ .  $N=21$  photons.

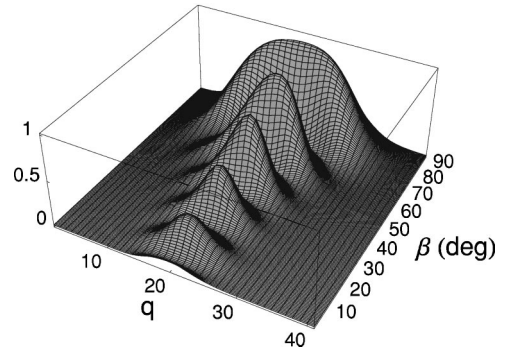


FIG. 8. Teleportation fidelity of an  $\alpha=3$  coherent state, using the quasi-EPR resource  $|\Psi_{10}^{(1)}(\beta)\rangle$ , versus the beam-splitter angle  $\beta$  and the number-sum measurement result  $q$ .  $N=20$  photons.

Before we plot  $F(q)$  for the two quasi-EPR states, we must address the question of the dependence of the fidelity on the measured value  $q$  of the number sum: this implies conditional teleportation even for an ideal relative-phase state ( $s_n^N = \text{const} \forall N, n$ ), which should not be.

Equation (40) has an upper bound, which can be found by using the Schwartz inequality:

$$\left| \sum_{k=k_0}^q |c_k|^2 s_{q-k}^N \right|^2 \leq \sum_{k=k_0}^q |c_k|^2 \sum_{k=k_0}^q |c_k|^2 |s_{q-k}^N|^2. \quad (41)$$

The fidelity is thus bounded by

$$F(q) \leq \sum_{k=k_0}^q |c_k|^2. \quad (42)$$

The physical meaning of this Schwartz inequality thus is the following: if  $k_0=0$  (i.e.,  $q \leq N$ , which is automatic for  $N \rightarrow \infty$ ) and  $q$  is very large compared to the spread of the target state (denoted by  $k_{max}$ , such that  $c_{k > k_{max}} = 0$ ), then the inequality (42) becomes

$$F(q) \leq \sum_{k=0}^{k_{max}} |c_k|^2 = 1, \quad (43)$$

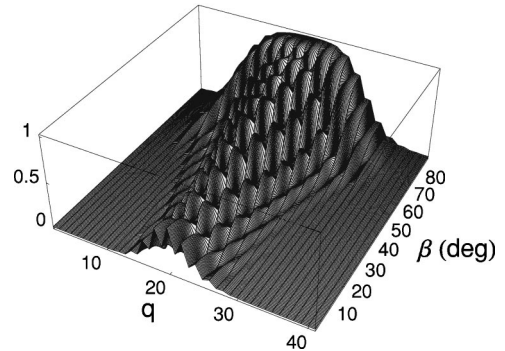


FIG. 9. Teleportation fidelity of an  $\alpha=3$  cat state, using the quasi-EPR resource  $|\Psi_{10}^{(1)}(\beta)\rangle$ , versus the beam-splitter angle  $\beta$  and the number-sum measurement result  $q$ .  $N=20$  photons.

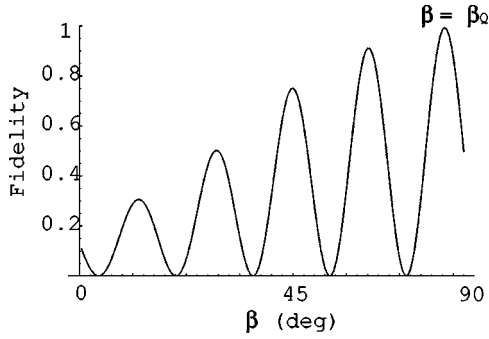


FIG. 10. Cross section of Fig. 8 for  $q=19$ . The peak value is 99.3% at  $\beta=\beta_Q=85.5^\circ$  ( $0.5^\circ$  steps). Note that  $\beta=90^\circ$  gives the classical fidelity value  $F=49.8\%$ , consistently with the results of CMM.

which corresponds to the fidelity of the ideal EPR resource. All of this is in fact ensured by maximal entanglement  $N \rightarrow \infty$ , and a finite  $k_{max}$ . When this is fulfilled, the probability that  $q < k_{max}$  becomes negligible and the teleportation becomes unconditional. Otherwise, one can have severe fidelity limitations, as in the worst case  $N < q < k_{max}$ :

$$F(q) \leq \sum_{k=q-N>0}^{q < k_{max}} |c_k|^2 < 1, \quad (44)$$

where the sum over the target state probability becomes truncated. Besides being lower, the fidelity will also depend strongly on the value of  $q$  (conditional teleportation).

In our computations,  $N$  is limited to a few tens of photons [calculating the state  $|\Psi_{10000}^{(1)}(\beta_Q)\rangle$  as in Fig. 4 is a lighter computational load than that of calculating  $|\Psi_{100}^{(2)}(\pi/2)\rangle$ ], making it difficult to approach  $N \rightarrow \infty$ . Therefore, the fidelity displayed in our figures has a remaining dependence on the value of  $q$  and the illusion of conditionality. The arguments above should have, however, convinced the reader that such is not the true physical situation for number-phase teleportation, which can be truly unconditional. As far as our numerical results are concerned, it is simple to see that, for a coherent target state  $|\alpha\rangle$  ( $\alpha$  real), the high-fidelity region is given by  $q \in [k_{max}, N - k_{min}]$  where  $N \gg k_{max}$ , i.e.,  $q \in [\alpha^2 + \alpha, N - \alpha^2 + \alpha]$ .

As mentioned earlier, we considered two different target states, a simple coherent state  $|\alpha\rangle_T$  and a ‘‘Schrödinger cat’’ state, the macroscopic, definitely nonclassical superposition

$$C(|\alpha\rangle_T + |-\alpha\rangle_T) = C \sum_{n=0}^{\infty} [1 + (-1)^n] \frac{\alpha^n}{\sqrt{n!}} |n\rangle_T, \quad (45)$$

where  $C = (2 + 2e^{-2|\alpha|^2})^{-1/2}$ . We calculated the teleportation fidelity  $F(q, \beta)$  for both these states with  $\alpha=3$ , using the two quasi-EPR states  $|\Psi_j^{(2)}(\beta)\rangle$  and  $|\Psi_j^{(1)}(\beta)\rangle$  [assuming the initial-entanglement-phase correction (32) is successfully applied by Bob] for the respective entanglement resources. Results are plotted in Figs. 6–9.

Once again, the limited spread in  $q$  of the high fidelity region is only due to the limitations of our computation. Had we been able to perform the simulations with a larger  $N$ , this effect would have disappeared. These teleportation protocols are unconditional. What is essential is that, in all cases, we obtain  $F \simeq 100\%$ . This occurs for  $\beta = \pi/2$  in Figs. 6 and 7 and for  $\beta = \beta_Q$  (see Fig. 10) in Figs. 8 and 9, thereby confirming our analysis that quasi-EPR states can be efficient teleportation resources.

#### IV. CONCLUSION

We have demonstrated that efficient quantum teleportation is indeed possible with close-to-maximally entangled states. We chose the simplest quasi-EPR states, which would be straightforward to generate should on-demand single-photon sources become an experimental reality. Note, however, that the same formalism applies equally well to second-quantized bosons in general, which opens possibilities for the use of trapped ions or Bose-Einstein condensates. We also investigated the use of generalized relative-phase states for quantum teleportation and showed that these states lead to serious complications if the basis ‘‘generator’’  $|\{\theta\}\rangle$  has a more complicated phase distribution than a simple phase offset. This therefore raises a question about the precise physical significance of the generalized relative-phase operator (29) of Ref. [27].

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