



Tracking gauge symmetry factorizability on intervals

Ngoc-Khanh Tran

Department of Physics, University of Virginia, 382 McCormick Road, Charlottesville, VA 22904-4714, USA

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Abstract

We track the gauge symmetry breaking pattern by boundary conditions on fifth and higher-dimensional intervals. It is found that, with Dirichlet–Neumann boundary conditions, the Kaluza–Klein decomposition in five-dimension for arbitrary gauge group can always be factorized into that for separate subsets of at most two gauge symmetries, and so is completely solvable. Accordingly, we present a simple and systematic geometric method to unambiguously identify the gauge breaking/mixing content by general set of Dirichlet–Neumann boundary conditions. We then formulate a limit theorem on gauge symmetry factorizability to recapitulate this interesting feature. Albeit the breaking/mixing, a particularly simple check of orthogonality and normalization of fields' modes in effective 4-dim picture is explicitly obtained. An interesting chained-mixing of gauge symmetries in higher dimensions by Dirichlet–Neumann boundary conditions is also explicitly constructed. This study has direct applications to higgsless/GUT model building.

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1. Introduction

One of the most beautiful piece that lies in the core of particle interaction standard model (SM) and elsewhere in condense matter physics is the concept of spontaneous symmetry breaking (SSB). In particle physics, this so-called Higgs mechanism is capable of generating masses for gauge bosons (i.e., breaking the corresponding gauge symmetries) and fermions in such a way that renormalizability and unitarity can be preserved. However, through extensive experimental search to date, the Higgs bosons which are the mechanism's most direct proponent have not been observed. In an interesting alternative approach with the presence of compact extra spatial dimensions [1], it is known that some gauge bosons of choice may or may not have massless

E-mail address: nt6b@galileo.phys.virginia.edu (N.-K. Tran).

mode in Kaluza–Klein (KK) decomposition, depending on their assigned boundary conditions (BC) and/or transformation properties under certain complementary discrete symmetries [2]. To be specific for later referencing, let us list here all three possible Dirichlet–Neumann BC configurations for a gauge field A living on an one-dimensional line segment $0 \leq y \leq \pi R$. All of these BCs can be constructed by some simple orbifold projection.

$$\begin{aligned}
 \text{DD configuration: } & A(y)|_0 = A(y)|_{\pi R} = 0 \Rightarrow A(y) = \sum_{m \in \mathbb{N}} A^{(m)} \sin \frac{my}{R}, \\
 \text{DN configuration: } & A(y)|_0 = \partial_y A(y)|_{\pi R} = 0 \Rightarrow A(y) = \sum_{m \in \mathbb{N}} A^{(m)} \sin \frac{(2m+1)y}{2R}, \\
 \text{NN configuration: } & \partial_y A(y)|_0 = \partial_y A(y)|_{\pi R} = 0 \Rightarrow A(y) = \sum_{m \in \mathbb{N}} A^{(m)} \cos \frac{my}{R}. \tag{1}
 \end{aligned}$$

Evidently, only in the NN configuration, gauge field possesses a non-trivial massless ($m = 0$) mode. In other words, gauge symmetry is preserved still only for boundary conditions at both end-points being of Neumann type. This so-called orbifold projection symmetry breaking (see e.g., [2]) has distinctive nature originated truly from boundary effects [3], that cannot be identified with the Higgs mechanism through the spontaneous breaking of background gauge field Wilson line. But again in this orbifold compactification approach, up to the possible absence of the zero mode, mass spectra of all gauge boson towers are identical, which makes it difficult to implement the $SU(2)_L \times U(1)_Y \rightarrow U(1)_{\text{QED}}$ electroweak breaking/mixing without reintroducing the Higgs below compactification scale.

To circumvent this obstacle, more general boundary conditions have been proposed for linear combinations of gauge fields rather than for only single ones [4] (earlier models using extra dimensional BCs to break the gauge symmetries, with or without traditionally full set of Higgs bosons, are described in [5,6]). As such, the corresponding compactification procedure generally grows out of the framework restricted by orbifold projections,¹ i.e., one is actually working with interval compactification. Since BCs are imposed on gauge fields’ linear combinations, it is plausible that symmetry breaking/preserving indeed can be obtained for mixtures of initial gauge groups. Then immediately come important queries on the systematic/exact connection, if such ever exists, between pattern of symmetry breaking/mixing and the BCs being employed. This kind of information is relevant, because after all, we definitely want to identify those BCs that can produce the desired symmetry mixings given beforehand. In the case of the Higgs SSB mechanism, it is the Goldstone theorem that can tell all about symmetry breaking pattern once the scalar field’s vacuum expectation value (VEV) configuration is picked. With that in mind, in this work we attempt to explore a similar connection between the gauge symmetry mixing on one-dimensional flat intervals and the boundary conditions applied on their ends. The investigation could be eventually generalized for larger number of extra dimensions and higher symmetry group ranks.

To have a taste of what specifically are going to be addressed later on, we first briefly review the original Higgs-less symmetry breaking $SU(2)_L \times SU(2)_R \times U(1)_{B-L} \rightarrow U(1)_{\text{QED}}$ on a fifth-dimensional interval $0 \leq y \leq \pi R$ [4,10,11] (see also [7]). As far as the breaking/mixing is concerned, already, this set-up is rich and equally mysterious, so it can serve as the debut

¹ Or else the equivalent set of required projectors, if such can be found, is quite complicated which makes the projection itself less practical in use.

of the current discussion. After the following Dirichlet and Neumann BCs² on some combinations of gauge bosons $A_L^{1,2,3}$ (of $SU(2)_L$, coupling g), $A_R^{1,2,3}$ (of $SU(2)_R$, coupling g) and B (of $U(1)_{B-L}$, coupling g') are imposed for neutral sector

$$\text{at } y = 0 \left\{ \begin{cases} \partial_y B_\mu = 0, \\ \partial_y (A_{L\mu}^3 + A_{R\mu}^3) = 0, \\ A_{L\mu}^3 - A_{R\mu}^3 = 0, \end{cases} \right. \quad \text{at } y = \pi R \left\{ \begin{cases} \partial_y A_{L\mu}^3 = 0, \\ \partial_y (g B_\mu + g' A_{R\mu}^3) = 0, \\ g' B_\mu - g A_{R\mu}^3 = 0, \end{cases} \right. \quad (2)$$

and for charged sector

$$\text{at } y = 0 \left\{ \begin{cases} \partial_y (A_{L\mu}^1 + A_{R\mu}^1) = 0, \\ \partial_y (A_{L\mu}^2 + A_{R\mu}^2) = 0, \\ A_{L\mu}^1 - A_{R\mu}^1 = 0, \\ A_{L\mu}^2 - A_{R\mu}^2 = 0, \end{cases} \right. \quad \text{at } y = \pi R \left\{ \begin{cases} \partial_y A_{L\mu}^1 = 0, \\ \partial_y A_{L\mu}^2 = 0, \\ A_{R\mu}^1 = 0, \\ A_{R\mu}^2 = 0, \end{cases} \right. \quad (3)$$

the solutions are found to be (up to normalization factors)

$$\left\{ \begin{aligned} B_\mu(x, y) &= g\gamma_\mu(x) + g' \sum_{m=1}^\infty \cos\left(\frac{\pm\phi}{\pi R} + \frac{m}{R}\right) Z_\mu^{(m)}(x), \\ A_\mu^{+3}(x, y) &\equiv \frac{A_{L\mu}^3 + A_{R\mu}^3}{\sqrt{2}} = g'\gamma_\mu(x) - g \sum_{m=1}^\infty \cos\left(\frac{\pm\phi}{\pi R} + \frac{m}{R}\right) Z_\mu^{(m)}(x), \\ A_\mu^{-3}(x, y) &\equiv \frac{A_{L\mu}^3 - A_{R\mu}^3}{\sqrt{2}} = g \sum_{m=1}^\infty \sin\left(\frac{\pm\phi}{\pi R} + \frac{m}{R}\right) Z_\mu^{(m)}(x) \\ &\quad (\text{where } \tan \phi \equiv \sqrt{g^2 + 2g'^2}/g) \end{aligned} \right. \quad (4)$$

$$\left\{ \begin{aligned} A_\mu^{+1,2}(x, y) &\equiv \frac{A_{L\mu}^{1,2} + A_{R\mu}^{1,2}}{\sqrt{2}} = \sum_{m=1}^\infty \cos\left(\frac{\pm 1}{4R} + \frac{m}{R}\right) W_\mu^{1,2(m)}(x), \\ A_\mu^{-1,2}(x, y) &\equiv \frac{A_{L\mu}^{1,2} - A_{R\mu}^{1,2}}{\sqrt{2}} = \sum_{m=1}^\infty \sin\left(\frac{\pm 1}{4R} + \frac{m}{R}\right) W_\mu^{1,2(m)}(x). \end{aligned} \right. \quad (5)$$

Beyond the obvious fact that the solutions (4), (5) satisfy the BCs (2), (3), it is tempting to gain more insights from this construction.

How comes only the intended $U(1)_{\text{QED}}$ (of massless photon $\gamma_\mu(x)$) is kept unbroken globally along the entire interval, following the different breakings $SU(2)_L \times SU(2)_R \times U(1)_{B-L} \rightarrow SU(2)_D \times U(1)_{B-L}$ and $SU(2)_L \times SU(2)_R \times U(1)_{B-L} \rightarrow SU(2)_L \times U(1)_Y$ made at $y = 0$ and $y = \pi R$, respectively. This is because, as indicated in (1), the combination of (2) allows uniquely a $U(1)$ gauge field to have Neumann BC at both end-points. A further question then is whether it is possible to track the symmetries that might be left unbroken/broken locally at any given point $y \neq 0, \pi R$ in the bulk, for the eventual sake of model building with brane-localized matter fields. Next, looking at the neutral sector alone, one begins with three gauge degrees of freedom ($A_L^3(x, y)$, $A_R^3(x, y)$, $B(x, y)$) in 5-dim, nevertheless it appears that one ends up with only one Kaluza–Klein (KK) tower ($\{Z^{(k)}(x)\}$) and eventually two different mass spectra in 4-dim, and a similar observation can be made for charged sector. This is seemingly because, as expected from differential equation theory, the BCs (2) (or (3)) render the ansatz that gives the mixed solutions and may constrain the number of free parameters in them. A further question then is

² These BCs are obtained in [4] for the limit of a brane-localized scalar infinite VEV. There are also BCs on the fifth components of gauge fields, which act to eliminate these components altogether in an appropriate gauge (Section 2.1).

how effectively BCs can be used to limit the number of 4-dim field towers, for the eventual sake of symmetry group rank reduction as in some classes of grand unified theory (GUT) breaking. Next, we also see from (4), (5) that the numbers of Neumann (and Dirichlet) BCs are equal on two ends of the interval, but these quantities do not represent the number of unbroken (and broken) gauges as naively expected from Eqs. (1). This is because, as mentioned above, the BCs induce a mixing between initial group's gauge fields, and what might matter for the KK decomposability is perhaps the equality of BCs' total number ($N + D$) on end-points. A further question then is how the mixing/breaking forms if, say, there are D Dirichlet, N Neumann BCs at $y = 0$, and D' , N' ones at $y = \pi R$ with a single constraint $D + N = D' + N'$, for the eventual sake of generalizing the construction.

In this paper we seek to answer these and discuss some other general questions on flat interval compactification with arbitrary gauge group and viable boundary conditions beyond Dirichlet and Neumann types. The recipe being exclusively used here to track the broken symmetry is to find out the gauge sector KK spectra under the given BCs, and identify their possible massless modes. In Section 2.1, we reconstruct in details the solutions (4), (5) using geometrical arguments. Essentially, each solution of a definite topological (KK) number can be regarded as a map between two boundary sets of constant fields. In this view, we cannot only track how the symmetry breaking propagates in the bulk, but also show that the expressions (4), (5) are not yet the most general solutions obeying the BCs (2), (3). Rather, photon and Z -boson can have different KK towers (this observation has been also made in [11]). In Section 2.2, we consider any large set of gauge fields living in a fifth-dimensional interval, which obey any Dirichlet–Neumann BC configurations. It is shown that the general gauge space can always be factorized, by virtue of Dirichlet–Neumann BC, into mutually orthogonal subspaces of no more than two dimensions each. Surprisingly, in this way, the initial general set-up is broken into smaller ones, which are solvable and indeed not more complicated than the original Higgs-less set-up described above. In Section 3 we analyze the orthogonality between massive gauge field modes as well as the compatibility between DN BCs and variational principle of action. This is important for the construction of 4-dim effective Lagrangian and the phenomenologies that follow it, such as perturbative unitarity in gauge bosons scattering [8]. Although a priori the fore-mentioned orthogonality is not apparent for system with mixed symmetry breaking under consideration, a posteriori both orthogonality and normalization of KK towers that survive in 4-dim are particularly transparent and simple. In Section 4 we give an explicit and systematic construction of gauge symmetry decomposition in higher dimension, which reveals an interesting and strict “chained entanglement” from one symmetry to the other by BCs. The possibility to non-trivially mix up to 2^d gauge symmetries in d extra dimensions is also demonstrated. In Section 5 we summarize the main results with some outlooks, and finally in Appendix A we prove a lemma on matrix factorizability, which is the basis for gauge space factorization presented in Section 2.2.

Throughout the presentation, we repeatedly invoke simple geometric interpretation/visualization to support our arguments. This approach is particularly suited for the flat space considered here. The method also appears very helpful for warped space investigation.

2. Gauge symmetry breaking along an 1-dim interval

In this section, we will consider a pure gauge set-up in $4 + 1$ space–time. The fifth dimension is finite, and the respective coordinate y runs from 0 to πR .

2.1. Propagation of symmetry breaking

Before tackling the original Higgs-less model described in the introduction section, let us work out first an apparently simpler problem with just two gauge degrees of freedom $N(x, y)$, $D(x, y)$. It turns out that this is all we practically need for the solution of the gauge multi-dimensional problem, as long as the Dirichlet–Neumann BCs are being employed. The action of this set-up is

$$S = \sum_{A=N,D} \int d^4x \int_0^{\pi R} dy \left(-\frac{1}{4} F_{\mu\nu}^A F^{A\mu\nu} - \frac{1}{2} (\partial_\mu A_5 - \partial_5 A_\mu) (\partial^\mu A^5 - \partial^5 A^\mu) + \dots + \mathcal{L}_{\text{GF}} \right), \tag{6}$$

where $F_{\mu\nu}^A \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ is the linearized field tensors, while the dots represent possible triple and quartic interactions for non-Abelian groups. As in gauge theory in 5-dim space–time, the 4-dim covariant gauge fixing term $\mathcal{L}_{\text{GF}} = \frac{1}{2\xi} (\partial_\mu A_\mu - \xi \partial_5 A_5)^2$ is chosen to render the cancellation of bilinear mixing between A_μ and A_5 . However, for the interval compactification, such cancellation is only up to a non-trivial surface term

$$\int_0^{\pi R} dy \left(-\frac{1}{2} F_{5\mu}^A F^{A5\mu} + \frac{1}{2\xi} (\partial_\mu A_\mu - \xi \partial_5 A_5) (\partial^\mu A^\mu - \xi \partial^5 A^5) \right) \supset (\partial_\mu A^\mu) A^5 \Big|_0^{\pi R} \tag{7}$$

which is not necessarily always zero for any A_5 configuration. Nevertheless, in unitary gauge $\xi \rightarrow \infty$ the derivative $\partial_5 A_5$ in \mathcal{L}_{GF} tends to zero, i.e., A_5 may have only massless mode. We further can choose either Dirichlet–Dirichlet or Dirichlet–Neumann BC configurations (1) for A_5 to suppress this zero mode. Thus in the case of 5-dim space–time, we can make all gauge field fifth components disappear by choice of gauge and BCs, and also eliminate the residual surface term (7). A more careful consideration [12] shows that in the unitary gauge, all non-zero KK modes $A_5^{(m)}$ indeed are “eaten” by longitudinal modes of massive bosons $A_\mu^{(m)}$. Thus hereafter, we can take $A_5 = 0$ identically.

We impose the following BCs on different set of gauge fields $\{N, D\}$ at $y = 0$ and $\{N', D'\}$ at $y = \pi R$ (with $0 \leq \phi < \pi$)

$$\begin{cases} \partial_y N_\mu(x, y)|_{y=0} = \partial_y N'_\mu(x, y)|_{y=\pi R} = 0, \\ D_\mu(x, y)|_{y=0} = D'_\mu(x, y)|_{y=\pi R} = 0, \end{cases} \quad \text{where} \quad \begin{pmatrix} N' \\ D' \end{pmatrix} \equiv \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} N \\ D \end{pmatrix}. \tag{8}$$

Obeying the linearized motion (Laplace’s) equations derived from the action (6), all gauge fields can be expressed in term of combination of trigonometric functions. Further, the BCs (8) allow us to look for N, D in the form: $N(x, y) = N(x) \cos My$, $D(x, y) = D(x) \sin My$. Then from the rewritten BCs on N', D' at $y = \pi R$

$$\begin{aligned} \partial_y N'_\mu(x, y)|_{y=\pi R} &= -M [N_\mu(x) \cos \phi \sin (M\pi R) + D_\mu(x) \sin \phi \cos (M\pi R)] = 0, \\ D'_\mu(x, y)|_{y=\pi R} &= N_\mu(x) \sin \phi \cos (M\pi R) + D_\mu(x) \cos \phi \sin (M\pi R) = 0 \end{aligned} \tag{9}$$

one finds

$$N(x) = \pm D(x) \quad \text{and} \quad \tan M\pi R = \mp \tan \phi \Rightarrow M = \frac{m}{R} \mp \frac{\phi}{\pi R} \quad (m \in Z). \tag{10}$$

There seem to be two mass spectra corresponding to two different signs in Eq. (10). However, as the integer m runs from minus to plus infinity, the two spectra can be merged into one (i.e., $|M_-^{(m)}| = |M_+^{(-m)}|$), which allows to write the general solution of fields in the following form

$$\begin{aligned}
 N(x, y) &= \sum_{m \in \mathbb{Z}} N^{(m)}(x) \cos\left(\frac{m}{R} + \frac{\phi}{\pi R}\right)y, \\
 D(x, y) &= \sum_{m \in \mathbb{Z}} -N^{(m)}(x) \sin\left(\frac{m}{R} + \frac{\phi}{\pi R}\right)y, \\
 N'(x, y) &= \sum_{m \in \mathbb{Z}} N^{(m)}(x) \cos\left(\left(\frac{m}{R} + \frac{\phi}{\pi R}\right)y - \phi\right), \\
 D'(x, y) &= \sum_{m \in \mathbb{Z}} -N^{(m)}(x) \sin\left(\left(\frac{m}{R} + \frac{\phi}{\pi R}\right)y - \phi\right).
 \end{aligned} \tag{11}$$

Indeed the BCs generate a relation (10) between the coefficients $N(x)$, $D(x)$, and thus produce only a “single” independent KK tower of gauge boson in 4-dim. However, for general ϕ , this tower (with $m \in \mathbb{Z}$) has twice as many 4-dim modes as does the tower (with $m \in \mathbb{N}$) given in Eq. (1) for $\phi = 0$ or $\frac{\pi}{2}$ ³. So properly speaking, the single tower $\{N^{(m)}(x)\}$ in Eq. (11) will be referred to as an extended tower hereafter. We choose to adopt this notation to simplify the writing and to eliminate the possible ambiguity over the double sign \pm in (10). We note that the lowest mass depends on ϕ : it is $\frac{\phi}{\pi R}$ (or $m = 0$) for $0 \leq \phi \leq \frac{\pi}{2}$, and $\frac{\pi - \phi}{\pi R}$ (or $m = -1$) for $\pi > \phi > \frac{\pi}{2}$. Finally, when $\phi = 0$ or $\frac{\pi}{2}$, we have $(N \sim N'; D \sim D')$ or $(N \sim D'; D \sim N')$ respectively, i.e., N decouples from D and each forms an independent 1-dim system with known BCs at both end-points given in (1). The gauge symmetry associated with N is unbroken.

This solution may be seen better geometrically. First, the definition (8) of the set $\{N', D'\}$ implies that it is rotated from the orthogonal set $\{N, D\}$ ⁴ by an angle ϕ , while the latter satisfying Dirichlet–Neumann BCs can be conveniently cast in the form

$$\begin{pmatrix} N(x, y) \\ D(x, y) \end{pmatrix} = \begin{pmatrix} \cos My & -\sin My \\ \sin My & \cos My \end{pmatrix} \begin{pmatrix} N(x) \\ 0 \end{pmatrix}. \tag{12}$$

Because this describes a $O(2)$ -rotation, it is also suggestive to pretend that, in term of value, the set $\{N(x, y), D(x, y)\}$ in turn is rotated from its original $\{N(x), 0\}$ (at $y = 0$) by the angle My as it propagates in the bulk. Next, the BCs (8) essentially signal that, again in term of values, $\{N', D'\}$ at $y = \pi R$, up to a sign, should be identical to $\{N, D\}$ at $y = 0$, which can be met when two above rotations precisely cancel one another. In expression, this is (see also Figs. 1, 2),

$$\begin{aligned}
 \text{for clockwise rotation:} & \quad M\pi R = \phi + m\pi \quad (m \in \mathbb{Z}), \\
 \text{for counter-clockwise rotation:} & \quad M\pi R = -\phi + m\pi \quad (m \in \mathbb{Z}).
 \end{aligned} \tag{13}$$

Clearly, KK mode number m is none other than the number of (half-)revolutions that the map from $\{N(x, y), D(x, y)\}$ at $y = 0$ to $\{N'(x, y), D'(x, y)\}$ at $y = \pi R$ wraps around, whether

³ In Eq. (1), because extra dimensional wavefunctions are identical (up to a sign) for a pair of negative and positive m , we can combine them and the resulting sum is limited to $m \in \mathbb{N}$. In contrast, because of the presence of a general twist angle ϕ , such consolidation is not possible for (11), thus $m \in \mathbb{Z}$ therein.

⁴ The quadratic terms of Lagrangian (6) are orthogonal in $\{N, D\}$ basis, so these can be seen as two orthogonal eigenvectors in gauge space.

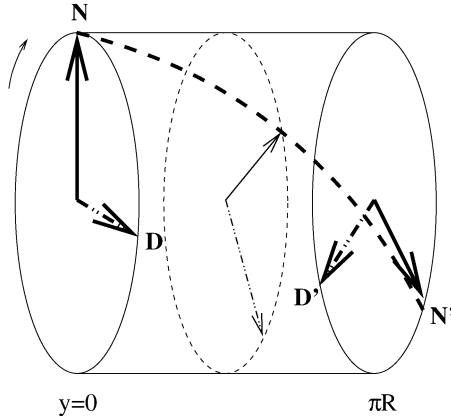


Fig. 1. Clockwise mapping $\{N, D\}$ at $y = 0$ to $\{N', D'\}$ at $y = \pi R$. The thick dashed line shows the bulk trajectory of Neumann direction in gauge symmetry space of some specific mode with map's winding number m .

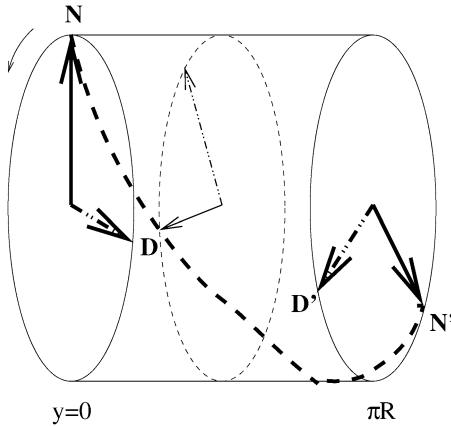


Fig. 2. Counter-clockwise mapping $\{N, D\}$ at $y = 0$ to $\{N', D'\}$ at $y = \pi R$. The thick dashed line shows the bulk trajectory of Neumann direction in gauge symmetry space of some specific mode with map's winding number m .

clockwise or counter-clockwise. Each value of m presents a solution for which the given BCs hold. But we again note that a rotation of m (half-)revolutions clockwise is equivalent to that of $-m$ ones counter-clockwise, so indeed the two spectra in (13) are the same. Hence the complete solution (11) consists of only a single (extended) KK towers $\{N^{(m)}(x)\}$ of 4-dim gauge fields. This geometrical construction further might shed light into the pattern of symmetry breaking/mixing at any point in the bulk. If we could limit the consideration to a particular solution (m fixed), it then follows from Eq. (12) that, in gauge space, the locally preserved symmetry direction (with vanishing field derivative) at point y would make an angle My with respect to vector field N , while the completely broken symmetry direction (with vanishing field) would make an angle $My + \frac{\pi}{2}$. Accordingly, if zero mode ($M = 0$) exists, then in that mode the locally preserved symmetry direction is unchanged ($My = 0$) for all y , i.e., that well-defined symmetry can be said to be unbroken throughout the bulk. This observation is in line with the effective 4-dim picture (where the fifth coordinate is integrated out leaving no mass term for zero mode), because zero mode wave function is constant on the entire interval. In either descriptions, the

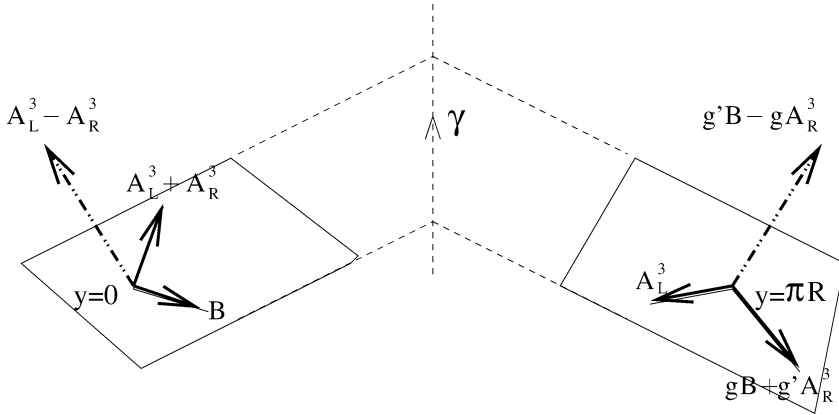


Fig. 3. Two 2-dim Neumann planes (defined at $y = 0$ and $y = \pi R$ respectively) intersect in gauge symmetry space along a 1-dim line, which is the $U(1)_{\text{QED}}$ unbroken symmetry direction with massless photon γ .

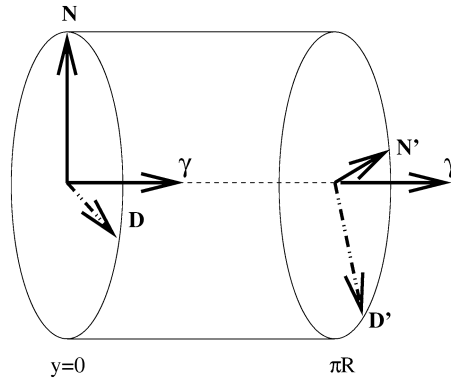


Fig. 4. Relative orientation of gauge fields at $y = 0$ and $y = \pi R$.

KK zero mode always is a reliable indicator of the associated preserved symmetry. However, because these rotation angles change with general m , when the comprehensive solution (summed over all m) (11) is undertaken in an intact 5-dim view, no where in the bulk a fixed direction can represent either locally preserved or completely broken symmetry, with the exception of two end-points.

We are now ready to embark on the original Higgs-less configuration. The neutral sector $\{A_{3L}, A_{3R}, B\}$ presents a 3-dim gauge space, with 1–2 (Dirichlet–Neumann) BCs on each ends. Since both Neumann and Dirichlet BCs are closed under additivity, at either end-points, we can have non-intersecting Dirichlet and Neumann subspaces, each contains only gauge fields satisfying the respective BC. In the Higgs-less 3-dim neutral gauge space (2), the 2-dim Neumann subspace $\{B, \frac{A_L^3 + A_R^3}{\sqrt{2}}\}$ at $y = 0$ and the other 2-dim Neumann subspace $\{A_L^3, \frac{gB + g'A_R^3}{\sqrt{g^2 + g'^2}}\}$ at $y = \pi R$ generally share one co-dimension, just exactly as two 2-dim planes intersect along a 1-dim line in 3-dim space (Fig. 3). That is, there exists one gauge vector field satisfying the Neumann BC at both end-points, i.e., by virtue of (1), the associated $U(1)_{\text{QED}}$ gauge symmetry is unbroken throughout the interval. It is straightforward to obtain this preserved symmetry direction γ ,

and then the orthogonal-to-it remaining 1–1 (Dirichlet–Neumann) spaces at end-points (Fig. 4). Note also that we generally cannot further factorize these residual 2-dim spaces, because the co-dimension of their constituent Dirichlet (or Neumann) subspaces, one at $y = 0$ and the other at $y = \pi R$, is zero, just exactly as two 1-dim lines intersect at a dimensionless point in 2-dim plane

$$\text{at } y = 0 \begin{cases} \partial_y \gamma_\mu|_{y=0} = 0, & \gamma \equiv \frac{g'(A_{3L} + A_{3R}) + gB}{\sqrt{g^2 + 2g'^2}}, \\ \partial_y N_\mu|_{y=0} = 0, & N \equiv \frac{g(A_{3L} + A_{3R}) - 2g'B}{\sqrt{2g^2 + 4g'^2}}, \\ D_\mu|_{y=0} = 0, & D \equiv \frac{A_{3R} - A_{3L}}{\sqrt{2}}, \end{cases}$$

$$\text{at } y = \pi R \begin{cases} \partial_y \gamma'_\mu|_{y=\pi R} = 0, & \gamma' \equiv \frac{g'(A_{3L} + A_{3R}) + gB}{\sqrt{g^2 + 2g'^2}}, \\ \partial_y N'_\mu|_{y=\pi R} = 0, & N' \equiv \frac{(g^2 + g'^2)A_{3L} - g'^2 A_{3R} - g g' B}{\sqrt{(g^2 + g'^2)(g^2 + 2g'^2)}}, \\ D'_\mu|_{y=\pi R} = 0, & D' \equiv \frac{gA_{3R} - g'B}{\sqrt{g^2 + g'^2}}. \end{cases} \quad (14)$$

Both $\{\gamma, N, D\}$ and $\{\gamma, N', D'\}$ sets are explicitly orthonormal, however $\{N', D'\}$ is rotated from $\{N, D\}$ by an angle ϕ (see Fig. 4 and Eq. (8)).

$$\begin{pmatrix} N' \\ D' \end{pmatrix} = \begin{pmatrix} \frac{g}{\sqrt{2g^2 + 2g'^2}} & -\frac{\sqrt{g^2 + 2g'^2}}{\sqrt{2g^2 + 2g'^2}} \\ \frac{\sqrt{g^2 + 2g'^2}}{\sqrt{2g^2 + 2g'^2}} & \frac{g}{\sqrt{2g^2 + 2g'^2}} \end{pmatrix} \begin{pmatrix} N \\ D \end{pmatrix} \Rightarrow \tan \phi = \frac{\sqrt{g^2 + 2g'^2}}{g}. \quad (15)$$

The solutions of the “decoupled” $\gamma(x, y)$ and the “entangled” pair $\{N, D\}$ now are readily given in (1) and (11)

$$\begin{aligned} \gamma(x, y) &= \gamma^{(0)}(x) + \sum_{m \neq 0} \cos \frac{my}{R} \gamma^{(m)}(x), \\ N(x, y) &= \sum_{m \in Z} Z^{(m)}(x) \cos \left(\frac{m}{R} + \frac{\phi}{\pi R} \right) y, \\ D(x, y) &= \sum_{m \in Z} -Z^{(m)}(x) \sin \left(\frac{m}{R} + \frac{\phi}{\pi R} \right) y, \end{aligned} \quad (16)$$

where ϕ is given in Eq. (15) and we have changed the notation from $N(x)$ in (11) to $Z(x)$ in (16) to facilitate the comparison with the earlier Higgs-less solution. From (16) one can obtain $\{B, A_{3L}, A_{3R}\}$ (again up to normalization factors) after using the definitions in (14)

$$\begin{aligned} B_\mu(x, y) &= g\gamma_\mu^{(0)}(x) + g \sum_{m=1}^{\infty} \gamma_\mu^{(m)}(x) \cos \frac{my}{R} - g' \sum_{m \in Z} Z_\mu^{(m)}(x) \cos \left(\frac{m}{R} + \frac{\phi}{\pi R} \right) y, \\ A_\mu^{+3}(x, y) &\equiv \frac{A_{L\mu}^3 + A_{R\mu}^3}{\sqrt{2}} = g' \gamma_\mu^{(0)}(x) + \sum_{m=1}^{\infty} g' \cos \frac{my}{R} \gamma_\mu^{(m)}(x) \\ &\quad + g \sum_{m \in Z} Z_\mu^{(m)}(x) \cos \left(\frac{m}{R} + \frac{\phi}{\pi R} \right) y, \\ A_\mu^{-3}(x, y) &\equiv \frac{A_{L\mu}^3 - A_{R\mu}^3}{\sqrt{2}} = g \sum_{m \in Z} Z_\mu^{(m)}(x) \sin \left(\frac{m}{R} + \frac{\phi}{\pi R} \right) y, \end{aligned} \quad (17)$$

where ϕ is given in (15). For the charged sector $\{A_L^1, A_L^2, A_R^1, A_R^2\}$, we first split it into two separate subspaces generated respectively by $\{A_L^{(1)}, A_R^{(1)}\}$ and $\{A_L^{(2)}, A_R^{(2)}\}$. Each of these 2D subspaces is just the familiar case of 1–1 Dirichlet–Neumann BCs (8) with $\phi = \frac{\pi}{4}$, so their solutions follow immediately from (11)

$$\begin{aligned}
 A_\mu^{+1,2}(x, y) &\equiv \frac{A_{L\mu}^{1,2} + A_{R\mu}^{1,2}}{\sqrt{2}} = \sum_{m \in Z} W_\mu^{1,2(m)}(x) \cos\left(\frac{m}{R} + \frac{1}{4R}\right)y, \\
 A_\mu^{-1,2}(x, y) &\equiv \frac{A_{L\mu}^{1,2} - A_{R\mu}^{1,2}}{\sqrt{2}} = \sum_{m \in Z} W_\mu^{1,2(m)}(x) \sin\left(\frac{m}{R} + \frac{1}{4R}\right)y.
 \end{aligned}
 \tag{18}$$

The expressions (17), (18) are more general than (4), (5) obtained first in [4]⁵, because here photon $\gamma^{(0)}(x)$ generally belongs to a distinctive (factorized) KK tower $\{\gamma^{(m)}(x)\}$ of distinctive mass spectra $\{\frac{m}{R}\}$. The mass $\frac{\phi}{\pi R}$ of Z-boson zero mode is always lighter than $\frac{1}{R}$ of photon’s first excited state. Thus the presence of photon KK tower does not spoil the ability to mimic the SM spectrum of this original Higgs-less model found in [4] at low energy scale. It needs, however, be taken into account in the processes involving real or virtual massive gauge bosons, for, e.g., to check the model’s unitarity. Further, we present (17), (18) in the single-spectrum form at the price of letting m have both negative and positive integral values.

We have just seen that, indeed the BCs can reduce half of the number of independent 4-dim gauge field towers⁶ in a simple set of two “entangled” gauge symmetries $\{N, D\}$. We next will explore this ability of Dirichlet–Neumann BCs, which are imposed again on the fifth interval’s two ends, but for arbitrary number of gauge symmetries.

2.2. Factorizability for arbitrary gauge dimensions

In this section, let us assume that, at $y = 0$, there be D and N gauge fields obeying the Dirichlet and Neumann BCs respectively. At $y = \pi R$, let those quantities be D' and N' . These gauge fields may represent Abelian (like $B(x, y)$) or non-Abelian (like $A_{L,R}^3$) symmetries. As we are interested in obtaining the KK decomposition of these fields, we just work first with the linearized (Laplace’s) motion equations.⁷ Therefore, within the stated purpose, there is no difference between Abelian and non-Abelian treatment. We also suppose that each of $\{N + D\}$ and $\{N' + D'\}$ be an orthonormal set of gauge vectors, just as naturally as we are working with gauge eigenstates that appear in the initial 5-dim Lagrangian. The original Higgs-less set-up obviously belongs to this class of construction. Finally, above BCs are perceived to act on four usual gauge field components, since in the 4 + 1 space–time, the fifth components can be made identically vanished by appropriate gauge’s choice and BCs on them.

In the case $D + N \neq D' + N'$, after eventual basis transformations, there will be fields with BCs being specified whether at only one of two end-points or more than once at a same end-point. The former situation generally leads to a non-quantized spectrum because of insufficient BCs, while the latter could lead to no spectrum at all because of redundant (and conflicting) BCs.

⁵ This fact has also been noted in [11]

⁶ Again note that the tower left in 4-dim picture is an “extended” tower.

⁷ Please note that the non-linear terms may recombine after this decomposition to produce the complex symmetry breaking patterns. The investigation of this interesting non-Abelian recombination mechanism however lies beyond the scope of this paper.

In this work we will not pursue either of these directions, though some particular BCs might be carefully selected to evade these shortcomings in the construction of standard KK decomposition.

In the remaining case $D + N = D' + N'$, we take, without loss of generality, $D > N, N'$. We can always assume further that each of the following pairs $(\{N\}, \{N'\})$, $(\{N\}, \{D'\})$, $(\{N'\}, \{D\})$ are non-intersecting, i.e., $\text{codim}(\{N\}, \{N'\}) = \text{codim}(\{N\}, \{D'\}) = \text{codim}(\{N'\}, \{D\}) = 0^8$, because if these are not zero, we can find (and solve) a number of gauge fields, each has one of three BCs configurations listed in Eq. (1). After we decouple these “solvable” fields from the set-up (just as we decouple $U(1)_{\text{QED}}$ gauge field from the Higgs-less neutral sector in previous section), we are left with only non-intersecting gauge boson sets. However, these sets are not necessarily mutually orthogonal in general (just as $\{N, D\}$ and $\{N', D'\}$ in Eq. (8) generally make an angle $\phi \neq \frac{\pi}{2}$).

We first create the union space $(\{N\} \cup \{N'\})$, which has $N + N'$ dimensions, since $\text{codim}(\{N\}, \{N'\}) = 0$. Next, we construct the space $\{D - N'\}$ of dimension $D - N'$ that complements $(\{N\} \cup \{N'\})$ in the entire gauge space, i.e.,

$$\{D - N'\} \cup (\{N\} \cup \{N'\}) = \{D + N\}.$$

Moreover, this complementary space $\{D - N'\}$ can always be built in such a way that it is mutually orthogonal to both $\{N\}$ and $\{N'\}$, that is any vector in $\{D - N'\}$ is orthogonal to all vectors in $\{N\}$ and $\{N'\}$, and vice versa.

As such, $\{D - N'\}$ necessarily is a subspace of $\{D\}$, because by construction, $\{D\}$ is the largest space that is mutually orthogonal to $\{N\}$ in the entire gauge space being spanned by the orthonormal set $\{N + D\}$. Similarly, $\{D - N'\}$ also necessarily forms a subspace of $\{D'\}$, which implies

$$\{D - N'\} \subset (\{D\} \cap \{D'\}).$$

In other words, each of $D - N'$ gauge fields that generate the space $\{D - N'\}$ has Dirichlet BC at both $y = 0$ (because it belongs to $\{D\}$) and $y = \pi R$ (because it belongs to $\{D'\}$). The KK decomposition of these $D - N'$ fields is given in (1), according to which all $D - N'$ associated symmetries are broken, and on the way we obtain $D - N'$ independent gauge boson massive KK towers in 4-dim of identical mass spectrum $\{\frac{m}{R}\}$ (with integer $m > 0$).

But this is not the end of the “decoupling” process. We are currently left with $N + N'$ gauge symmetries, among them N' and N orthonormal fields satisfy respectively the Dirichlet and Neumann BCs at $y = 0$. At $y = \pi R$ those numbers are N (for Dirichlet’s) and N' (for Neumann’s). For clarity, let us denote the corresponding spaces generated by these gauge vector sets as $\{N'_{D,0}\}$, $\{N_{N,0}\}$, $\{N_{D,\pi R}\}$ and $\{N'_{N,\pi R}\}$, where the subscripts are informative about the type and position of BCs, while the main capital letters are about the number of fields as always. Again, without loss of generality we take $N' > N$ and repeat the above steps to further identify and isolate the “decoupled” sectors. We construct the space $\{N' - N\}$ of $N' - N$ dimensions that complements the union space $(\{N_{N,0}\} \cup \{N_{D,\pi R}\})$ of $2N$ dimensions (because $\text{codim}(\{N_{N,0}\}, \{N_{D,\pi R}\}) = 0$) in the (now) entire gauge space $\{N + N'\}$, i.e.,

$$\{N' - N\} \cup (\{N_{N,0}\} \cup \{N_{D,\pi R}\}) = \{N + N'\}.$$

Since $\{N' - N\}$ are chosen to be mutually orthogonal to both $\{N_{N,0}\}$ and $\{N_{D,\pi R}\}$, it must be a subspace of the intersection $(\{N'_{D,0}\} \cap \{N'_{N,\pi R}\})$. These $N' - N$ fields then have Dirichlet BC at

⁸ Notation: $\text{codim}(\{N\}, \{D'\})$ denotes the co-dimension of Neumann subspace generated by $\{N\}$ gauge fields and the Dirichlet subspace generated by $\{D'\}$ gauge fields in gauge space.

$y = 0$ and Neumann's at $y = \pi R$. In the result all the $N' - N$ associated gauge symmetries are necessarily broken. Their KK decomposition (1) consists of $N' - N$ independent massive gauge boson towers in 4-dim of identical mass spectrum $\frac{2m+1}{2R}$.

At this point, after two successive (but similar) reduction processes, there are still $2N$ “coupled” gauge symmetries by the following BCs distribution (in the above notation): $\{N_{D,0}\}$, $\{N_{N,0}\}$, $\{N_{D,\pi R}\}$ and $\{N_{N,\pi R}\}$. Each of $(\{N_{D,0}\} \cup \{N_{N,0}\})$ and $(\{N_{D,\pi R}\} \cup \{N_{N,\pi R}\})$ forms an orthonormal basis of the (now) entire $2N$ -dim gauge space and so one is necessarily related to the other by some general orthogonal “BC-defining” matrix $O(2N)$ (i.e., the matrix given in the BCs definition like Eq. (8)). In practice, knowing fields on which BCs are imposed (e.g., Eq. (2)) we can determine the BC-defining matrix (e.g., Eq. (15)). Repeating the same simple argument once more likely would not help to advance the symmetry “disentanglement”, if there ever is such possibility for the $2N$ -dim space at hands. There exists, however, a convincing clue that indeed this gauge space is subject to further “disentanglement”. Because fields obeying Dirichlet (or Neumann) BCs form a closed set under their linear combination transformations, along the decoupling process, we have the freedom to perform four independent $O(N)$ rotations separately on the sets $\{N_{D,0}\}$, $\{N_{N,0}\}$, $\{N_{D,\pi R}\}$ and $\{N_{N,\pi R}\}$. This implies that, among the $\frac{2N(2N-1)}{2}$ parameters of the initial BC-defining matrix $O(2N)$, we can eliminate $4\frac{N(N-1)}{2}$ ones by four $O(N)$ basis transformations. We are left with N physical parameters (say $\{\phi_1, \dots, \phi_N\}$) which exactly equals the number of parameters in a set of N $O(2)$ matrices. We then vaguely expect that the general $2N$ -symmetry set could decouple at least into N independent elementary sets, each consists of no more than two gauge dimensions $\{N, D\}$ discussed in details in Section 2.1. Interestingly, this expectation turns out to be a rigorous result obtained through a lemma introduced in Appendix A, which confirms the diagonal factorizability of any general $O(2N)$ matrix into N $O(2)$ blocks by four $O(N)$ independent rotations. In this manner, for a general (non-zero) set $\{\phi_1, \dots, \phi_N\}$, the initial $2N$ symmetries are all broken by BC compactification, in the result of which there are N distinctive gauge field KK towers in 4-dim of distinctive mass spectra $\{\frac{m}{R} + \frac{\phi_i}{\pi R}\}$ (with $i = 1$ to N). Again, this factorization can be visualized geometrically as follows. Given at the onset two arbitrary orthonormal eigenbases $(\{N_{D,0} \cup N_{N,0}\})$ and $(\{N_{D,\pi R} \cup N_{N,\pi R}\})$ in $2N$ -dim spaces, one can rearrange internally the basis vectors within four N -dim subsets $(\{N_{D,0}\}, \{N_{N,0}\}, \{N_{D,\pi R}\}, \{N_{N,\pi R}\})$ to divide the initial space into N mutually orthogonal 2-dim planes. Each of these planes can be identified by two alternative bases made of the fore-mentioned rearranged vectors. For i th plane, the two bases are off one from the other by angle ϕ_i , which characterizes the corresponding KK mass spectrum. In the special case $\phi_i = 0$ for some $i \in [0, N]$, the corresponding i th plane can be further decoupled into 2 independent directions, one of which generally represents a preserved symmetry as shown in Section 2.1.

From the proof presented in Appendix A, we in particular note two important points. First, the factorization is unique once the “BC-defining” matrix $O(2N)$ is given, i.e., the KK decomposition spectra are unambiguously determined for the set of the initial $2N$ Dirichlet–Neumann BCs. Second, the fact that N 2-dim planes obtained in the factorization process are *mutually orthogonal* is an indispensable condition for them to develop N associated *independent* spectra. Or putting in another order, this statement means: any subspace (with known BCs) being mutually orthogonal to the rest can be factored out, and becomes a separate problem subject to KK decomposition on its own right.

Let us now recapitulate our discussion so far on the pattern of symmetry breaking by BCs and the resulting spectra in the following theorem.

Table 1
Symmetry breaking on interval with Dirichlet–Neumann boundary conditions

Total initial symmetries	$S = D + N = D' + N'$		
BC distribution (# Dirichlet, # Neumann)	$(D, N)_{y=0}$ & $(D', N')_{y=\pi R}$		
Assumption (without generality loss)	$D > N' > N$ $\text{codim}(\{N\}, \{N'\}) = \text{codim}(\{N\}, \{D'\}) = \text{codim}(\{N'\}, \{D\}) = 0$		
Sector's symmetry	Completely broken	Completely broken	Mixed broken & unbroken
Sector's dimensions	$D - N'$	$N' - N$	$2N$
BC type ($y = 0$)–($y = \pi R$)	Dirichlet–Dirichlet	Dirichlet–Neumann	Mixed
Mass spectrum	$\frac{m}{R}$ ($m \neq 0$)	$\frac{2m+1}{2R}$	$\frac{m}{R} + \frac{\phi_i}{\pi R}$
# 4-dim independent towers	$D - N'$	$N' - N$	$\geq N$

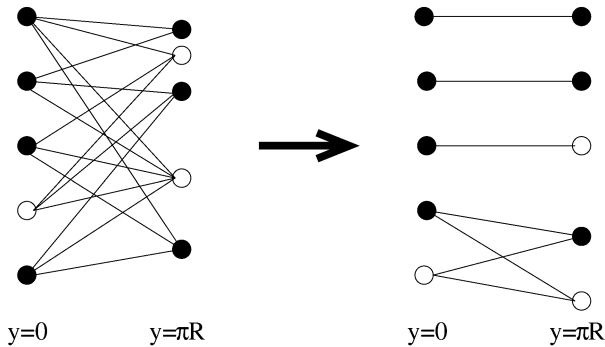


Fig. 5. Gauge symmetry sets before (left scheme) and after (right scheme) some basis transformations (or factorization). Shaded and empty dots represent gauge fields with Neumann and Dirichlet BC respectively. Connected fields are related (“entangled”) in their definition. Within each scheme, fields being in the same horizontal level are identical.

Limit theorem on gauge symmetry factorizability: For a set of S gauge symmetries on a fifth-dimensional interval subject to totally $2S$ Dirichlet and Neumann BCs, (i) the total number of independent (extended) gauge field towers (and equally of associated mass spectra) in 4-dim is no less than $\frac{S}{2}$, (ii) the number of unbroken symmetries (and equally of mass spectra with zero mode) is generally given by the number of mutually orthogonal gauge fields that have Neumann BCs at both end-points.

The more quantitative description of this theorem is presented in Table 1 (see also footnote 7). The specific case $N = 1, D = 4, N' = 2, D' = 3$ is illustrated in Fig. 5. While the above theorem literally identifies the number of 4-dim towers into which the fields’ modes are grouped as the result of compactification, it is really about the possibility to “entangle” the symmetries by Dirichlet–Neumann BCs. In case of a single extra dimension, one can at most combine two initial gauge directions to produce a single (though extended) tower of vector fields, which we see in 4-dim. Evidently, this “hybrid” tower receives contribution from both initial 5-dim gauge fields, and this observation is crucial for the modes’ orthogonality and normalization as we discuss next. When more extra dimensions are allowed, we will see in Section 4 that more fields can be “entangled”.

3. From KK mode orthogonalities to effective Lagrangian in 4-dim

In this section we will construct the 4-dim effective Lagrangian that results from integrating out the fifth dimension. The integration gives rise to surface terms, that a priori do not automatically vanish because fields with defined BCs are not the same on the two boundaries. Rather, the nullifying of these terms sets constraints on “qualified” BCs through the general variational principle on the action. In Section 3.1 we examine the KK mode orthogonality from both on-set BCs and KK decomposition. In Section 3.2 we perform the explicit normalization of extra dimensional wavefunctions and thus obtain the effective gauge boson self-couplings.

3.1. Orthogonality between KK modes

We first consider the (real) KK wavefunctions $g^{(k)}(y)$, $g^{(l)}(y)$ corresponding to modes k, l of an initial gauge field $G(x, y)$ in 5-dim ($\partial_y^2 g^{(k)} = m_k^2 g^{(k)}$; $\partial_y^2 g^{(l)} = m_l^2 g^{(l)}$). From the integration by parts

$$\int_0^{\pi R} g^{(k)} \partial_y^2 g^{(l)} dy = g^{(k)} \partial_y g^{(l)} \Big|_0^{\pi R} - (\partial_y g^{(k)}) g^{(l)} \Big|_0^{\pi R} + \int_0^{\pi R} (\partial_y^2 g^{(k)}) g^{(l)} dy \tag{19}$$

follows the wavefunction overlap

$$(m_l^2 - m_k^2) \int_0^{\pi R} g^{(k)} g^{(l)} dy = [g^{(k)} \partial_y g^{(l)} - (\partial_y g^{(k)}) g^{(l)}] \Big|_0^{\pi R}. \tag{20}$$

For $k = l$ this equality trivially holds, but for $k \neq l$, if we impose the orthogonality between $\{g^{(k)}\}$, this is translated into a condition on the BCs

$$[g^{(k)} (\partial_y g^{(l)}) - (\partial_y g^{(k)}) g^{(l)}] \Big|_0^{\pi R} = 0. \tag{21}$$

In practice, one may have BCs on either G or $\partial_y G$ (but not both) at an end-point, or else their BCs are not known at the other end-point (like the Dirichlet–Neumann set-up of previous section), so a priori the orthogonality condition (21) is not evidently met by itself in interval compactification. Actually, the subtle cancellation holds only collectively between all fields in each “entangled” sector as we will see now. It is by the way worthwhile to note that (21) is equivalent to the completeness relation given in [4] (because both have same root in the hermiticity of $\hat{\partial}_y^2$ in a finite interval), which is required to annihilate the E^4 term in longitudinal gauge boson scattering amplitude (see also [14,15]). We will come back to this assertion in the next section.

From the previous analysis, any gauge symmetry set with Dirichlet–Neumann BCs can be factorized into decoupled subsets of one gauge dimension and “entangled” subsets of two dimensions. The decoupled subsets (e.g., $\gamma(x, y)$ of neutral sector in Eq. (14)) have as BCs one of three configurations listed in Eq. (1), so the orthogonality for decoupled sector is obviously satisfied. As a direct check for the KK decomposition (1), we indeed have

$$\int_0^{\pi R} \sin \frac{ky}{R} \sin \frac{ly}{R} dy \sim \int_0^{\pi R} \sin \frac{(2k+1)y}{2R} \sin \frac{(2l+1)y}{R} dy \sim \int_0^{\pi R} \cos \frac{ky}{R} \cos \frac{ly}{R} dy \sim \delta_{k,l} \tag{22}$$

because $k, l \in N$. If one repeats this direct check for KK decomposition of field $D(x, y)$ (or $N(x, y)$) give in Eq. (11)

$$\int_0^{\pi R} \sin\left(\frac{k}{R} + \frac{\phi}{\pi R}\right)y \sin\left(\frac{l}{R} + \frac{\phi}{\pi R}\right)y dy \neq 0,$$

$$\int_0^{\pi R} \cos\left(\frac{k}{R} + \frac{\phi}{\pi R}\right)y \cos\left(\frac{l}{R} + \frac{\phi}{\pi R}\right)y dy \neq 0 \tag{23}$$

one finds apparently KK modes of $D(x, y)$ (or $N(x, y)$) are not orthogonal to one the others. This can also be explained by the fact that orthogonality condition (21) does not hold for each of $D(x, y)$ and $N(x, y)$ as suspected. This seems to be a little counter-intuitive to the standard KK decomposition procedure. Either in continuous [12] or latticized [16–22] (see also [23]) extra dimension models with presence of some brane/bulk scalar VEVs, the orthogonality of gauge bosons modes is quasi-automatic because these are eigenmodes of a hermitian matrix to be diagonalized. In the current situation, as $D(x, y)$ and $N(x, y)$ belong to an entangled sector and specially share the same 4-dim tower $\{N^{(m)}(x)\}$, we should not have looked at each of them individually. Rather, let us check the KK mode orthogonality and simultaneously derive the effective Lagrangian of the sector $\{D(x, y), N(x, y)\}$ as a whole. We again concentrate on the abelianized part of the action and set all field fifth components to zero, leaving non-quadratic interactions as well as normalization factors for the next section’s effective couplings discussion.

$$\begin{aligned} \mathcal{L}^{(4)} &= \sum_{k,l \in Z} \int_0^{\pi R} dy \left(-\frac{1}{4} F_{\mu\nu}^{N(k)} F^{N(l)\mu\nu} - \frac{1}{4} F_{\mu\nu}^{D(k)} F^{D(l)\mu\nu} - \frac{1}{2} F_{5\mu}^{N(k)} F^{N(l)5\mu} \right. \\ &\quad \left. - \frac{1}{2} F_{5\mu}^{D(k)} F^{D(l)5\mu} \right) \\ &= \sum_{k,l \in Z} \left(-\frac{1}{4} F_{\mu\nu}^{(k)}(x) F^{(l)\mu\nu}(x) + \frac{1}{2} \left(\frac{k\pi + \phi}{\pi R} \right) \left(\frac{l\pi + \phi}{\pi R} \right) N_{\mu}^{(k)}(x) N^{(l)\mu}(x) \right) \\ &\quad \times \int_0^{\pi R} dy \left(\cos \frac{(k\pi + \phi)y}{\pi R} \cos \frac{(l\pi + \phi)y}{\pi R} + \sin \frac{(k\pi + \phi)y}{\pi R} \sin \frac{(l\pi + \phi)y}{\pi R} \right), \tag{24} \end{aligned}$$

where the 4-dim linearized field tensor $F_{\mu\nu}^{(k)}(x) \equiv \partial_{\mu} N_{\nu}^{(k)}(x) - \partial_{\nu} N_{\mu}^{(k)}(x)$ and we have used the explicit solution (11). Evidently, the integration over the fifth coordinate produces a Kronecker’s delta $\delta_{k,l}$, thus enforces the orthogonality between modes $N_{\mu}^{(k)}(x)$ in 4-dim. From Eqs. (23), (24) it is clear that this orthogonality can be achieved only when all 5-dim fields of an entangled sector are taken into consideration, because indeed the sector’s unique 4-dim tower is part of all of them. Such visualization might also serve in favor of the 4-dim “extended” tower viewpoint adopted for each independent sector.

Alternatively, we now can see more directly the role of BC on the KK modes’ orthogonality from another perspective. This time we work with the Dirichlet–Neumann BCs, but not the explicit decomposition solution. When the extra dimension is integrated out to obtain the effective Lagrangian, we note that only the cross-terms containing the derivative over fifth coordinate ($\partial_y N_{\mu} \in F_{5\mu}$) can generate surface contribution to the 4-dim action

$$\begin{aligned}
 S^{(4)} &\supset -\frac{1}{2} \int_0^{\pi R} dy (F_{5\mu}^N(x, y) F^{N5\mu}(x, y) + F_{5\mu}^D(x, y) F^{D5\mu}(x, y)) \\
 &= \frac{1}{2} \left[\{N_\mu(x, y) D_\mu(x, y)\} \left\{ \begin{matrix} \partial_y N_\mu(x, y) \\ \partial_y D_\mu(x, y) \end{matrix} \right\} \Big|_0^{\pi R} \right. \\
 &\quad \left. - \int_0^{\pi R} dy \{N_\mu(x, y) D_\mu(x, y)\} \left\{ \begin{matrix} \partial_y^2 N_\mu(x, y) \\ \partial_y^2 D_\mu(x, y) \end{matrix} \right\} \right]. \tag{25}
 \end{aligned}$$

Because all gauge fields obeying the Laplace’s equation in the bulk, we can replace ∂_y^2 in the second term of (25) by some 4-dim squared mass M^2 , and then this term’s orthogonality follows in exactly the same way as it does for the pure 4-dim kinetic terms considered earlier. In contrast, the surface term could present some unwanted contribution (which might spoil the mode orthogonality), and in model-building practice should be set to zero by appropriate choice of BCs. Indeed, with Dirichlet–Neumann BC, the surface term of effective action (25) vanishes identically at both end points and thus it leaves no destructive effect on the orthogonality of KK decomposition in 4-dim.

3.2. Normalization and effective gauge boson self-couplings

Let us now complete the construction of effective Lagrangian by normalizing the extra dimensional wave functions. As we have already seen, the mode diagonality concerns all symmetries altogether within each independent sector, and normalization should be proceeded in the coherent way. That is, one should not normalize N and D modes separately basing on the expansion (11).

For the decoupled sector (one of three configurations in Eq. (1)), the normalization factor is $\sqrt{2/\pi R}$ for all modes, with the only exception (being $\sqrt{1/\pi R}$) of the massless mode in the Neumann–Neumann configuration. For the entangled sector (8), the normalization is given by the integration given in the sector’s effective Lagrangian (24) (putting $l = k$)

$$\left[\int_0^{\pi R} dy \left(\cos \frac{(k\pi + \phi)y}{\pi R} \cos \frac{(l\pi + \phi)y}{\pi R} + \sin \frac{(k\pi + \phi)y}{\pi R} \sin \frac{(l\pi + \phi)y}{\pi R} \right) \right]_{(k=l)}^{-1/2} = \frac{1}{\sqrt{\pi R}}. \tag{26}$$

It is interesting to note that the normalization factor is independent of both mode number and twist angle ϕ . As an illustration, we are now ready to write down the complete (normalized) KK expansions (17), (18) of the original Higgs-less model

$$\begin{aligned}
 B(x, y) &= \left[g\gamma^{(0)}(x) + g\sqrt{2} \sum_{m \neq 0} \gamma^{(m)}(x) \cos \frac{my}{R} - g'\sqrt{2} \sum_{m \in \mathbb{Z}} Z^{(m)}(x) \cos \left(\frac{m}{R} + \frac{\phi}{\pi R} \right) y \right] \\
 &\quad \times [\sqrt{\pi R} \sqrt{g^2 + 2g'^2}]^{-1},
 \end{aligned}$$

$$\begin{aligned}
 A_{L,R}^3(x, y) &= \left[g'\gamma^{(0)}(x) + g'\sqrt{2} \sum_{m \neq 0} \gamma^{(m)}(x) \cos \frac{my}{R} \right. \\
 &\quad \left. + \sqrt{g^2 + g'^2} \sum_{m \in \mathbb{Z}} Z^{(m)}(x) \cos \left\{ \left(\frac{m}{R} + \frac{\phi}{\pi R} \right) y \mp \phi \right\} \right] [\sqrt{\pi R} \sqrt{g^2 + 2g'^2}]^{-1},
 \end{aligned}$$

$$\begin{aligned}
A_L^{1,2}(x, y) &= \frac{\sum_{m \in Z} W^{1,2(m)}(x) \cos\left\{\left(\frac{m}{R} + \frac{1}{4R}\right)y - \frac{\pi}{4}\right\}}{\sqrt{\pi R}}, \\
A_R^{1,2}(x, y) &= \frac{\sum_{m \in Z} W^{1,2(m)}(x) \sin\left\{\left(\frac{m}{R} + \frac{1}{4R}\right)y - \frac{\pi}{4}\right\}}{\sqrt{\pi R}}
\end{aligned} \tag{27}$$

and the respective effective Lagrangian (with the $SU(2)$ complete field tensor $F_{MN}^a(x, y) = \partial_M A_N^a(x, y) - \partial_N A_M^a(x, y) + g\epsilon^{abc} A_M^b(x, y) A_N^c(x, y)$ and 5-space-time Lorentz indices $M, N = 0, \dots, 4$)

$$\begin{aligned}
\mathcal{L}^{(4)} &= \frac{-1}{4} \int_0^{\pi R} \left(F_{MN}^B F^{BMN} + \sum_{a=1}^3 F_{MN}^{La} F^{LaMN} + \sum_{a=1}^3 F_{MN}^{Ra} F^{RaMN} \right) dy \\
&= \left(\sum_{A=\gamma, Z, W^1, W^2} \sum_m \frac{-1}{4} (\partial_\mu A_\nu^{(m)} - \partial_\nu A_\mu^{(m)}) (\partial^\mu A^{(m)\nu} - \partial^\nu A^{(m)\mu}) \right. \\
&\quad \left. + \frac{1}{2} M_A^{(m)} A_\mu^{(m)} A^{(m)\mu} \right) \\
&\quad - \frac{g \sin \phi \cos \phi \sqrt{g^2 + g'^2}}{\sqrt{\pi R (g^2 + 2g'^2)}} \left(\frac{1}{\phi} - \frac{1}{2\phi - \pi} - \frac{1}{2\phi + \pi} \right) (\partial_\mu W_\nu^{1(0)}) W^{2(0)\mu} Z^{(0)\nu} \\
&\quad - \frac{gg'}{\sqrt{\pi R (g^2 + 2g'^2)}} (\partial_\mu W_\nu^{1(0)}) W^{2(0)\mu} \gamma^{(0)\nu} + \dots,
\end{aligned} \tag{28}$$

where the dots denote the triple and quartic gauge interactions, which can be found by plugging the expansions (27) into the effective Lagrangian. Whereas, we have explicitly specified the zero mode triple interactions $W^{1(0)} W^{2(0)} Z^{(0)}$ and $W^{1(0)} W^{2(0)} \gamma^{(0)}$ in Eq. (28). If $(W^{1(0)} \mp i W^{2(0)})/\sqrt{2}$ are to be identified with SM charged bosons W^\pm , then the ratio of these couplings presents the SM weak mixing angle

$$\frac{g \sin \phi \cos \phi \sqrt{g^2 + g'^2}}{g'} \left(\frac{1}{\phi} - \frac{1}{2\phi - \pi} - \frac{1}{2\phi + \pi} \right) = \cot \theta_W. \tag{29}$$

Since $\tan \phi = \frac{\sqrt{g^2 + 2g'^2}}{g}$, this gives a relation between 5-dim couplings g, g' . We can also straightforwardly compute other effective couplings between gauge field higher modes. In particular, the self-interaction of $W^{1(m)} W^{2(n)} \gamma^{(p)}$ is found to be

$$g_{W^{1(m)} W^{2(n)} \gamma^{(p)}} = g' \sqrt{2} \int_0^{\pi R} dy \cos \frac{py}{R} \cos \frac{(m-n)y}{R} \sim \delta_{p+m-n, 0} + \delta_{p-m+n, 0}. \tag{30}$$

This effective coupling signals an “accidental” fifth-momentum conservation, a result of the precise canceling contribution from the initial $SU(2)_L$ and $SU(2)_R$ to W towers. This raises a possibility (which is highly model-dependent) that, due to symmetry entanglement by BCs, perturbative unitarity could be achieved in some carefully constructed models when the criterion (21) is not held separately for each initial field.

Though matter fields (Higgs scalars and fermions) can be introduced at either end-points or throughout the bulk, it is not easy to make this original Higgs-less set-up in 5-dim flat space-time cope simultaneously with other SM constraints [4]. More realistic versions of this model

take into account the warped geometry factor [11] or more extra dimensions [9] (see also next section). To encompass these directions, a more general study of gauge space factorizability in interval compactification is underway and will be presented elsewhere [13].

4. Perspectives on gauge space symmetry mixing in higher dimensions

By now, the consideration of Section 2.2 has led to the conclusion that in 5-dim space, one cannot “entangle” more than two gauge symmetries by however-sophisticated Dirichlet–Neumann BCs. A natural question to ask is whether this limitation on gauge mixing by BCs can be somewhat lifted in higher-dim space (a.k.a. can we mix more symmetries with more dimensions). The final answer is affirmative, and in this section we will demonstrate this general trend through an explicit, but systematic and simple construction of symmetry mixing in any dimensions.

4.1. Few simplest extensions

Let us first consider the simplest extension of the above 5-dim set-up: we now work with 3-dim gauge problem $\{G_1, G_2, G_3\}$ in the 6-dim space (i.e., two extra dimensions) with the coordinates denoted as (x_μ, y_1, y_2) . The extra dimensions are all finite with length πR (i.e., $0 \leq y_1, y_2 \leq \pi R$). There exist different ways to define the boundary conditions on this extra dimensional square. Here again, the observation of Section 2.2 can help simplify our choice. Along 1-dim interval, say along Oy_1 (i.e., $y_2 = 0$), one can truly mix only two out of three gauge bosons $\{G_1, G_2, G_3\}$. This prompts us to define the now-familiar, but indeed most general, Dirichlet–Neumann BCs on the following field sets along Oy_1 direction (see also Eq. (8))

along Oy_1 :

$$\begin{cases} \partial_{y_1} G_1|_{y_1=0} = \partial_{y_1} G'_1|_{y_1=\pi R} = 0, \\ G_2|_{y_1=0} = G'_2|_{y_1=\pi R} = 0, \end{cases} \quad \text{where} \quad \begin{pmatrix} G'_1 \\ G'_2 \\ G'_3 \end{pmatrix} \equiv \begin{pmatrix} \cos \phi_1 & -\sin \phi_1 & 0 \\ \sin \phi_1 & \cos \phi_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} G_1 \\ G_2 \\ G_3 \end{pmatrix}. \quad (31)$$

In the above equation, $\{G'_1, G'_2, G'_3\}$ is just an auxiliary field set (an alternative basis of the same 3-dim gauge space) which serves to specify the BCs, and the gauge field G_3 represents the “disentangled” symmetry along Oy , for which we do not need to identify the BC. Below, we will explain this point after we obtain the decomposition solution for the gauge fields. Similarly, another set of BCs can be imposed along Oy_2 in the same logical pattern

along Oy_2 :

$$\begin{cases} \partial_{y_2} G_2|_{y_2=0} = \partial_{y_2} G''_2|_{y_2=\pi R} = 0, \\ G_3|_{y_2=0} = G''_3|_{y_2=\pi R} = 0, \end{cases} \quad \text{where} \quad \begin{pmatrix} G''_1 \\ G''_2 \\ G''_3 \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \phi_2 & -\sin \phi_2 \\ 0 & \sin \phi_2 & \cos \phi_2 \end{pmatrix} \begin{pmatrix} G_1 \\ G_2 \\ G_3 \end{pmatrix}. \quad (32)$$

The decomposition solution of each of (31), (32) separately has been readily obtained in Eq. (11). However, to accommodate both (31), (32) we need to add in certain multiplicative factors $h(y_1)$ and $l(y_2)$

$$\begin{cases} G_1 \sim \cos\left(\frac{m_1}{R} + \frac{\phi_1}{\pi R}\right)y_1 l(y_2), \\ G_2 \sim -\sin\left(\frac{m_1}{R} + \frac{\phi_1}{\pi R}\right)y_1 l(y_2), \end{cases} \quad (33)$$

$$\begin{cases} G_2 \sim h(y_1) \cos\left(\frac{m_2}{R} + \frac{\phi_2}{\pi R}\right)y_2, \\ G_3 \sim -h(y_1) \sin\left(\frac{m_2}{R} + \frac{\phi_2}{\pi R}\right)y_2, \end{cases} \quad (34)$$

where m_1, m_2 are all integers. After consolidating G_2 in (33) and (34) we can solve for $h(y_1)$, $l(y_2)$ and unambiguously obtain the general decomposition of the gauge fields from our 6-dim set-up

$$\begin{aligned} G_1(x_\mu, y_1, y_2) &= \sum_{m_1, m_2 \in \mathbb{Z}} g^{(m_1, m_2)}(x_\mu) \cos\left(\left[\frac{m_1}{R} + \frac{\phi_1}{\pi R}\right]y_1\right) \cos\left(\left[\frac{m_2}{R} + \frac{\phi_2}{\pi R}\right]y_2\right), \\ G_2(x_\mu, y_1, y_2) &= - \sum_{m_1, m_2 \in \mathbb{Z}} g^{(m_1, m_2)}(x_\mu) \sin\left(\left[\frac{m_1}{R} + \frac{\phi_1}{\pi R}\right]y_1\right) \cos\left(\left[\frac{m_2}{R} + \frac{\phi_2}{\pi R}\right]y_2\right), \\ G_3(x_\mu, y_1, y_2) &= \sum_{m_1, m_2 \in \mathbb{Z}} g^{(m_1, m_2)}(x_\mu) \sin\left(\left[\frac{m_1}{R} + \frac{\phi_1}{\pi R}\right]y_1\right) \sin\left(\left[\frac{m_2}{R} + \frac{\phi_2}{\pi R}\right]y_2\right). \end{aligned} \quad (35)$$

There are a number of interesting observations we can make by looking at the above decomposition. First, the general solution (35) clearly indicates that the composite KK tower $g^{(m_1, m_2)}(x)$ of gauge boson in 4-dim is synthesized from all three initial gauge fields G_1, G_2, G_3 in 6-dim. That is, in other words, in higher dimensions we can indeed mix/break more than two symmetries by general boundary conditions. Second, when $\phi_y = \phi_z = \phi_u = 0$, it comes with no surprise that in this model, G_2, G_3 do not have non-trivial zero mode, and we expect that the associated symmetries are completely broken in 4-dim effective picture. This is because in Eqs. (31), (32), G_2, G_3 do have Dirichlet BC at certain boundaries.

Third and most amazingly, in the transition from the disconnected decompositions $G_1 - G_2$ and $G_2 - G_3$ (33), (34) to the composite solutions $G_1 - G_2 - G_3$ (35), gauge symmetries G_1 and G_3 have been “entangled”, though we did not start out with any explicit BCs that mix these two symmetries. Mathematically, this “chained-entanglement” is manifested by the fact that the identification of G_2 in (33), (34) unambiguously determines both coefficient $h(y_1)$ and $l(y_2)$. Had we imposed another DN boundary condition of the type (31) or (32) on $G_1 - G_3$ sector, we would have ended up generally with no decomposition solution at all by the redundancy of BCs on the underlying (differential) motion equation of fields (see the 3-extra dim set-up below). In a more visual description, the first BC entangles G_1 with G_2 , the second BC entangle G_2 with G_3 , and in the result all three are automatically inter-connected.

Now we can also see better the reason why in the above construction we did not specify the BCs for G_3 along y_1 , and G_1 along y_2 , respectively in (31) and (32). Explicitly, Eq. (32) states the BCs on G_2, G_3 only along y_2 , but *implicitly*, Eq. (32) also forcefully requires that the analytic dependences of G_2 and G_3 wave functions on coordinate y_1 be identical, otherwise BCs (32) simply do not hold. (In Eq. (34), we used the same expression $h(y_1)$ in both $G_2(y_1, y_2)$ and $G_3(y_1, y_2)$ just to enforce this implicit effect of (32) boundary conditions). Furthermore, Eq. (31) explicitly imposes BC on G_2 along y_1 . Thus, through the virtue of G_2 , the combination of implicit effect of BC (32) and explicit effect of BC (31) can unambiguously determine the analytical behavior of G_3 along y_1 . Similarly we can determine the analytical dependence of G_1 on y_2 without needs to impose the corresponding explicit boundary condition. The *implicit* power of BCs (31), (32) obviously is effective only for differential equations on multiple variables, and this gives rise to the “chained-entanglement” feature observed here in the higher-dimensional space.

This striking feature of boundary conditions on the gauge symmetry can be employed further to determine the extent of gauge symmetry mixing in any number of space–time dimensions. But

before doing that, let us quickly discuss a 7-dim set-up (i.e., 3-extra dim) to clearly show that gauge mixing/entanglement may indeed be destroyed by over-redundant BCs. Assume that we impose the following D-N BCs on 3-gauge symmetries in 7-dim

along Oy_1 :

$$\begin{cases} \partial_{y_1} G_2|_{y_1=0} = \partial_{y_1} G'_2|_{y_1=\pi R} = 0, \\ G_3|_{y_1=0} = G'_3|_{y_1=\pi R} = 0, \end{cases} \quad \text{where } \begin{pmatrix} G'_1 \\ G'_2 \\ G'_3 \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \phi_1 & -\sin \phi_1 \\ 0 & \sin \phi_1 & \cos \phi_1 \end{pmatrix} \begin{pmatrix} G_1 \\ G_2 \\ G_3 \end{pmatrix}, \quad (36)$$

along Oy_2 :

$$\begin{cases} G_1|_{y_2=0} = G''_1|_{y_2=\pi R} = 0, \\ \partial_{y_2} G_3|_{y_2=0} = \partial_{y_2} G''_3|_{y_2=\pi R} = 0, \end{cases} \quad \text{where } \begin{pmatrix} G''_1 \\ G''_2 \\ G''_3 \end{pmatrix} \equiv \begin{pmatrix} \cos \phi_2 & 0 & \sin \phi_2 \\ 0 & 1 & 0 \\ -\sin \phi_2 & 0 & \cos \phi_2 \end{pmatrix} \begin{pmatrix} G_1 \\ G_2 \\ G_3 \end{pmatrix}, \quad (37)$$

along Oy_3 :

$$\begin{cases} \partial_{y_3} G_1|_{y_3=0} = \partial_{y_3} G'''_1|_{y_3=\pi R} = 0, \\ G_2|_{y_3=0} = G'''_2|_{y_3=\pi R} = 0, \end{cases} \quad \text{where } \begin{pmatrix} G'''_1 \\ G'''_2 \\ G'''_3 \end{pmatrix} \equiv \begin{pmatrix} \cos \phi_3 & -\sin \phi_3 & 0 \\ \sin \phi_3 & \cos \phi_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} G_1 \\ G_2 \\ G_3 \end{pmatrix}, \quad (38)$$

The separated solutions corresponding to each of the above BC set are

$$\begin{cases} G_2 \sim \cos\left(\frac{m_1}{R} + \frac{\phi_1}{\pi R}\right)y_1, \\ G_3 \sim -\sin\left(\frac{m_1}{R} + \frac{\phi_1}{\pi R}\right)y_1, \\ G_1 \sim \cos\left(\frac{m_3}{R} + \frac{\phi_3}{\pi R}\right)y_3, \\ G_2 \sim -\sin\left(\frac{m_3}{R} + \frac{\phi_3}{\pi R}\right)y_3. \end{cases} \quad \begin{cases} G_1 \sim -\sin\left(\frac{m_2}{R} + \frac{\phi_2}{\pi R}\right)y_2, \\ G_3 \sim \cos\left(\frac{m_2}{R} + \frac{\phi_2}{\pi R}\right)y_2, \end{cases}$$

It is not difficult to convince ourselves that, indeed the above three separated solutions cannot be consolidated into a single one of the type (35), i.e., no decomposition solution exists if we use altogether three DN BC sets (36)–(38). Again, this is because BC (36) entangles G_2 with G_3 , (37) entangles G_3 with G_1 , and at this point G_1 and G_2 should have been already mixed by virtue of G_3 . Since we here pressed on to impose (38) which once more directly mixes G_1 and G_2 , this new BC apparently is redundant and in conflict with the effect of (36), (37).

The Dirichlet–Neumann BCs of the types (31), (32) have been argued in literature to be consistent with action’s variational principle (e.g., [4] for 5-dim, [9] for higher dimensions) by showing that the non-redundant DN BC set is sufficient to nullify all surface terms in the variation of the action δS . Consequently, non-redundant DN BC set can be properly obtained by minimizing the action (and there may be more than one qualified/alternative non-redundant BC sets depending on the action parameters’ values [3,4]). As thus, by applying the variational principle on action, we generally would not expect to be able to generate *all* BCs (e.g., (36)–(38)) in redundant set because those BCs are more than enough/needed (i.e., redundant) to nullify the surface terms. The reason here again is the implicit power of each BC on all coordinates in higher dimensions, that creates conflicting effects on gauge fields wave functions if all BCs of redundant set are simultaneously imposed.

4.2. General perspectives: S symmetries mixed in d extra dimensions

We are now ready for the general case of S symmetry degrees of freedom in $(d + 4)$ space–time dimensions. It may be very useful if we visualize each symmetry as a dot, and each set of 1D–1N BC (e.g., (36) or (37) or (38)) that entangles two symmetries as a link connecting the two corresponding dots. In this picture, to avoid the above redundancy of boundary conditions, there should be only one possible (whether direct or indirect) path to travel between any two dots (along the established links) (Fig. 6). If there are two distinct paths connecting dot G_i to dot G_j , this just means gauge fields G_i and G_j are entangled by two independent ways using the given sets of BCs (Fig. 7). From the discussion of the last section, this “multiple entanglement” would destroy the mixed decomposition of gauge fields by the redundancy of those boundary conditions. According to this “non-redundancy” rule, in a graph with S dots, we can establish $(S - 1)$ “non-redundant” links, each represents a 1D–1N BC set in our convention. With these $(S - 1)$ links, we also note that no dot (or a set of dots) is disconnected from the rest in the graph, which keeps the non-factorizability of the decomposition solution of the entire gauge fields’ set.

Next we embed this “chained entanglement” into the $(d + 4)$ -dim space–time (i.e., d extra dimensions). Each 1D–1N BC set along 1 extra dimension, say O_{y_1} , will contribute two possible factors $(\cos(\frac{m_1}{R} + \frac{\phi_1}{\pi R})y_1$ and $\sin(\frac{m_1}{R} + \frac{\phi_1}{\pi R})y_1$) to the gauge fields’ wave functions. So in case of d extra dimensions, the composite KK tower can accommodate/mix up to 2^d initial gauge symmetries $\{G_1, \dots, G_{2^d}\}$. More specifically, the corresponding decomposition solution looks

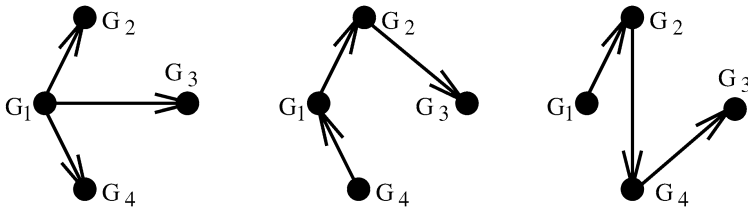


Fig. 6. Three of possible “non-redundant” ways to “entangle” the four gauge symmetries G_1, G_2, G_3, G_4 . Each link (or arrow) represents a set of 1D–1N boundary condition that mixes the two corresponding connected symmetries. Non-redundancy means that there is only one path connecting any two symmetries. The number of non-redundant links equals $(S - 1)$ in graph with S dots.

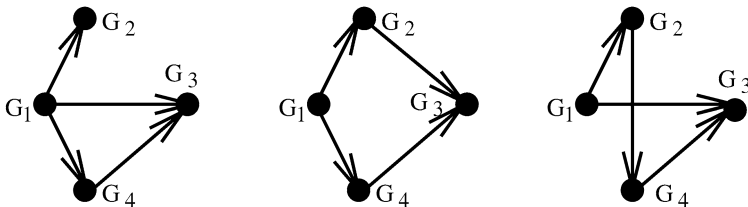


Fig. 7. Three of possible “redundant” ways to “entangle” the four gauge symmetries G_1, G_2, G_3, G_4 . Redundancy means that there are more than one distinct paths connecting two symmetries. For, e.g., in the first graph, two distinct paths connecting G_1 and G_3 are: $(G_1 - G_3)$ and $(G_1 - G_4 - G_3)$. The redundancy will destroy the decomposition solution of the gauge fields.

like

$$\begin{aligned}
 &G_1(x_\mu, y_1, \dots, y_d) \\
 &= \sum_{m_1, \dots, m_d \in \mathbb{Z}} g^{(m_1, \dots, m_d)}(x_\mu) \cos\left(\frac{m_1}{R} + \frac{\phi_1}{\pi R}\right) y_1 \cdots \cos\left(\frac{m_d}{R} + \frac{\phi_d}{\pi R}\right) y_d, \\
 &\vdots \\
 &G_{2^d}(x_\mu, y_1, \dots, y_d) \\
 &= \sum_{m_1, \dots, m_d \in \mathbb{Z}} g^{(m_1, \dots, m_d)}(x_\mu) \sin\left(\frac{m_1}{R} + \frac{\phi_1}{\pi R}\right) y_1 \cdots \sin\left(\frac{m_d}{R} + \frac{\phi_d}{\pi R}\right) y_d. \tag{39}
 \end{aligned}$$

Combining the observations obtained in this section, we finally come to the following conclusion:

In the space with d extra finite dimensions, it is possible to mix up to $S \leq 2^d$ gauge symmetries by imposing $(S - 1)$ sets of 1D–1N boundary conditions.

There is one more subtle but interesting point underlying this conclusion that we need to address. That is, can all sets of BCs employed be totally independent? In general, the answer is negative, simply because if they were all different, then the “entangled” KK spectrum would be characterized by up to $(2^d - 1)$ different twist angles $\{\phi_i\}$. But as we have seen in Eq. (39) that the most extensive “entangled” spectrum can contain only d twist angles, so clearly BC sets cannot be all different in general. Below is a more rigorous analysis of this issue.

In the case of two extra dimension, we now have some basis to expect that we may mix four gauge symmetries $\{G_1, G_2, G_3, G_4\}$ (but not only three as we did in Eq. (35)). Here we attempt to do this with three different sets of 1N–1D BCs (i.e., $\phi_{11} \neq \phi_{12} \neq \phi_2$ a priori)

$$\begin{aligned}
 &G_{1,N}[O^{(y_1, \phi_{11})}(2)]G_{2,D}, & G_{3,N}[O^{(y_1, \phi_{12})}(2)]G_{4,D}, \\
 &G_{1,N}[O^{(y_2, \phi_2)}(2)]G_{3,D}, \tag{40}
 \end{aligned}$$

where the short-hand notation $G_{1,N}[O^{(y_1, \phi_{11})}(2)]G_{2,D}$ denotes the familiar 1D–1N BC set imposed on G_1, G_2 along y_1 -direction with twist angle ϕ_{11} . The decomposition solution of each of these BC sets separately is

$$\begin{cases} G_1 \sim \cos\left(\frac{m}{R} + \frac{\phi_{11}}{\pi R}\right)y_1, \\ G_2 \sim -\sin\left(\frac{m}{R} + \frac{\phi_{11}}{\pi R}\right)y_1, \end{cases} \quad \begin{cases} G_3 \sim \cos\left(\frac{m}{R} + \frac{\phi_{12}}{\pi R}\right)y_1, \\ G_4 \sim -\sin\left(\frac{m}{R} + \frac{\phi_{12}}{\pi R}\right)y_1, \end{cases} \\
 \begin{cases} G_1 \sim \cos\left(\frac{m}{R} + \frac{\phi_2}{\pi R}\right)y_2, \\ G_3 \sim -\sin\left(\frac{m}{R} + \frac{\phi_2}{\pi R}\right)y_2. \end{cases}
 \end{cases}$$

Clearly, when $\phi_{11} \neq \phi_{12}$ it is impossible to consolidate the above three solutions into a unified one. In fact, we can only do this if $\phi_{11} = \phi_{12} \equiv \phi_1$, and the unified KK decomposition satisfying all three BC sets (40) reads

$$\begin{aligned}
 G_1(x_\mu, y_1, y_2) &= \sum_{m_1, m_2 \in \mathbb{Z}} g^{(m_1, m_2)}(x_\mu) \cos\left(\left[\frac{m_1}{R} + \frac{\phi_1}{\pi R}\right]y_1\right) \cos\left(\left[\frac{m_2}{R} + \frac{\phi_2}{\pi R}\right]y_2\right), \\
 G_2(x_\mu, y_1, y_2) &= - \sum_{m_1, m_2 \in \mathbb{Z}} g^{(m_1, m_2)}(x_\mu) \sin\left(\left[\frac{m_1}{R} + \frac{\phi_1}{\pi R}\right]y_1\right) \cos\left(\left[\frac{m_2}{R} + \frac{\phi_2}{\pi R}\right]y_2\right),
 \end{aligned}$$

$$\begin{aligned}
G_3(x_\mu, y_1, y_2) &= - \sum_{m_1, m_2 \in \mathbb{Z}} g^{(m_1, m_2)}(x_\mu) \cos\left(\left[\frac{m_1}{R} + \frac{\phi_1}{\pi R}\right]y_1\right) \sin\left(\left[\frac{m_2}{R} + \frac{\phi_2}{\pi R}\right]y_2\right), \\
G_4(x_\mu, y_1, y_2) &= \sum_{m_1, m_2 \in \mathbb{Z}} g^{(m_1, m_2)}(x_\mu) \sin\left(\left[\frac{m_1}{R} + \frac{\phi_1}{\pi R}\right]y_1\right) \sin\left(\left[\frac{m_2}{R} + \frac{\phi_2}{\pi R}\right]y_2\right). \quad (41)
\end{aligned}$$

Indeed, all four initial symmetries $\{G_1, G_2, G_3, G_4\}$ was non-trivially mixed in 6-dim to produce a single (composite) KK gauge field tower $g^{(m_1, m_2)}(x_\mu)$ in 4-dim. The analysis also points out an important feature: for the existence of a unified decomposition solution, only one BC twist angle can be associated with each extra dimension, though a same set of 1D–1N BCs can be imposed repeatedly on more than one set of fields. (In above, e.g., when $\phi_{11} = \phi_{12}$, the same BC set indeed was imposed on both $(G_1 - G_2)$ and $(G_3 - G_4)$.)

Finally, we note that the construction of gauge symmetry mixing in higher dimension presented here is not the only possible or most-efficient one, because there are more possible ways to define BCs (e.g., on surface instead of on a direction) with more dimensions. This construction however has quite simple and interesting entanglement structure because it is built on the limitation of symmetry mixing in one extra dimension found in Section 2.2. The construction further yields exact solutions (35), (41), which are sufficient for our current primary goal to show the explicit non-trivial “chained entanglement” of up to 2^d gauge symmetries in $(d + 4)$ -dim space–time. A general treatment of gauge fields decomposition in higher dimensions by any imposition patterns of Dirichlet–Neumann BCs however lies beyond the scope of this work.

5. Conclusion

In this work we have investigated the symmetry breaking of arbitrary gauge group by general Dirichlet–Neumann boundary conditions imposed at two ends of a fifth-dimensional interval. Such boundary conditions induce the factorization of the initial gauge symmetry space into subspaces of only one or two dimensions. Those subspaces are mutually orthogonal to one the other, and thus each gives an independent KK tower of gauge fields in 4-dim.

In gauge space, each unbroken symmetry direction form such one-dimensional “decoupled” subspace, which is necessarily represented by a vector field having Neumann boundary conditions at both interval’s ends. Each two-dimensional “entangled” subspace is characterized by a twist angle ϕ , which gives the relative orientation between field sets with defined BCs at end-points. This angle determines the mass spectrum of the associated 4-dim tower, and can be tuned to produce phenomenological light gauge boson as its lowest mode. Further, in the 4-dim effective picture, while this angle has no effect on the modes’ diagonality and normalization, it controls the non-Abelian triple and quartic gauge boson self-interaction.

In an intact 5-dim viewpoint, though the group symmetry breaking scheme makes sense locally at end-points, it does not at any other points in the bulk. Matter fields with definite charges under end-groups then can be placed at end-branes to calibrate brane/bulk coupling ratio, or else the fifth coordinate can be integrated out to produce a full 4-dim effective picture.

In higher dimension, the imposition of Dirichlet–Neumann BCs on pairs of gauge fields along different finite dimensions creates an very interesting “chained entanglement” of the symmetries. This entanglement is also very strict in its nature that it tolerates just a minimum (non-redundant) sets of 1D–1N BCs. We hope to come back with a more general analysis of gauge symmetry breaking on higher and non-flat intervals in future.

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Appendix A. A lemma on matrix factorization

In this appendix, we will prove a lemma, which constitutes the formal ground for the factorizability of gauge symmetry space by Dirichlet–Neumann presented in Section 2.2.

Lemma. Any general orthogonal matrix $O(2N)$ can always be brought into the block-diagonal form (of N $O(2)$ -blocks) by four separate and independent $O(N)$ rotations, two of which act on the right and the other two act on the left of the original $O(2N)$ matrix.

In expression, this means, given a general $2N \times 2N$ orthogonal matrix

$$O \equiv \left(\begin{array}{c|c} M & N \\ \hline P & Q \end{array} \right),$$

one can always find four independent $O(N)$ matrices A, B, C, D for which the following equality holds

$$\left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array} \right) \left(\begin{array}{c|c} M & N \\ \hline P & Q \end{array} \right) \left(\begin{array}{c|c} C & 0 \\ \hline 0 & D \end{array} \right) = \left(\begin{array}{ccc|ccc} c_1 & 0 & 0 & s_1 & 0 & 0 \\ 0 & c_2 & 0 & 0 & s_2 & 0 \\ & & \ddots & & & \\ 0 & 0 & c_N & 0 & 0 & s_N \\ \hline -s_1 & 0 & 0 & c_1 & 0 & 0 \\ 0 & -s_2 & 0 & 0 & c_2 & 0 \\ & & \ddots & & & \\ 0 & 0 & -s_N & 0 & 0 & c_N \end{array} \right), \tag{A.1}$$

where $c_i \equiv \cos \phi_i, s_i \equiv \sin \phi_i$. Evidently, the matrix on the right-hand side has the block-diagonal form (of N $O(2)$ -blocks) after some permutations of columns (and rows).

Proof. Given the matrix $O \in O(2N)$, our objective is to identify four matrices $A, B, C, D \in O(N)$ that satisfy Eq. (A.1). Because O is an orthogonal matrix, $OO^T = O^T O = \mathbf{1}_{2N \times 2N}$ which implies relations between M, N, P, Q

$$\begin{aligned} \left(\begin{array}{c|c} MM^T + NN^T & MP^T + NQ^T \\ \hline PM^T + QN^T & PP^T + QQ^T \end{array} \right) &= \left(\begin{array}{c|c} \mathbf{1}_{N \times N} & 0 \\ \hline 0 & \mathbf{1}_{N \times N} \end{array} \right) \\ &= \left(\begin{array}{c|c} M^T M + P^T P & M^T N + P^T Q \\ \hline N^T M + Q^T P & N^T N + Q^T Q \end{array} \right). \end{aligned} \tag{A.2}$$

Let us define A and C^T as the $N \times N$ orthogonal matrices that diagonalize the symmetric matrices MM^T and $M^T M$ respectively, i.e.,

$$A(MM^T)A^T = \begin{pmatrix} c_1^2 & & \\ & \ddots & \\ & & c_N^2 \end{pmatrix} = C^T(M^T M)C. \tag{A.3}$$

Because $MM^T + NN^T = \mathbf{1}_{N \times N} = M^T M + P^T P$ (see Eq. (A.2)), the above definitions of A and C also imply

$$A(NN^T)A^T = \begin{pmatrix} s_1^2 & & \\ & \ddots & \\ & & s_N^2 \end{pmatrix} = C^T(P^T P)C. \tag{A.4}$$

Next, let us define B and D^T as the orthogonal matrices that diagonalize PP^T and $N^T N$ respectively (note that KK^T and $K^T K$ have same set of eigenvalues for any square real matrix K)

$$B(PP^T)B^T = \begin{pmatrix} s_1^2 & & \\ & \ddots & \\ & & s_N^2 \end{pmatrix} = D^T(N^T N)D. \tag{A.5}$$

And since $QQ^T + PP^T = \mathbf{1}_{N \times N} = Q^T Q + N^T N$, in place of (A.4) now we have

$$B(QQ^T)B^T = \begin{pmatrix} c_1^2 & & \\ & \ddots & \\ & & c_N^2 \end{pmatrix} = D^T(Q^T Q)D. \tag{A.6}$$

We first consider the non-degenerate case where $c_i \neq c_j$ for every $i \neq j$. Then the matrices A, C in (A.3) and B, D in (A.5) are uniquely defined. In consequence, from (A.3) follows that the general real $N \times N$ matrix M is diagonalized by A and C , i.e., $(AMC) \sim \text{diag}(c_1, c_2, \dots, c_N)$. Similar conclusions can be drawn from combination of (A.4) and (A.5), or (A.3) and (A.6), so altogether we have

$$|AMC| = \begin{pmatrix} |c_1| & & \\ & \ddots & \\ & & |c_N| \end{pmatrix} = |BQD|, \tag{A.7}$$

$$|AND| = \begin{pmatrix} |s_1| & & \\ & \ddots & \\ & & |s_N| \end{pmatrix} = |BPC|. \tag{A.8}$$

Next, it follows from the relation $MP^T + NQ^T = 0$ (see (A.2)) that

$$\begin{aligned} 0 &= A(MP^T + NQ^T)B^T = (AMC)(C^T P^T B^T) + (AND)(D^T Q^T B^T) \Rightarrow \\ &(AMC)(BPC)^T + (AND)(BQD)^T = (AMC)(BPC) + (AND)(BQD) = 0 \end{aligned} \tag{A.9}$$

because both (BPC) and (BQD) are diagonal (A.7), (A.8). After explicitly expanding the matrix product on the left hand size of (A.1)

$$\left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array} \right) \left(\begin{array}{c|c} M & N \\ \hline P & Q \end{array} \right) \left(\begin{array}{c|c} C & 0 \\ \hline 0 & D \end{array} \right) = \left(\begin{array}{c|c} AMC & AND \\ \hline BPC & BQD \end{array} \right) \tag{A.10}$$

and using (A.7)–(A.9) we indeed obtain the equality (A.1). \square

For the degenerate case (where $c_i = c_j$ for some $i \neq j$), all orthogonal matrices A, B, C, D can only be determined up to some sub-rotation within each degenerate subspace. Due to this ambiguity, (A.3) does not necessarily imply that (AMC) be a diagonal matrix (and same for

(BQD), (AND), (BPC)). However, as the degeneracy pattern is identical for (A.3)–(A.6) we can still pick simultaneously a set of A, B, C, D , for which all four matrices (AMC), (BQD), (AND), (BPC) are diagonal. In the result, in this case too, the equality (A.1) is satisfied.

In the Dirichlet–Neumann physical system considered in Section 2.2, the $2N \times 2N$ orthogonal matrix O presents the relation between two sets of $2N$ gauge fields with defined BCs at $y = 0$ and $y = \pi R$, respectively. The $N \times N$ orthogonal matrices A, B, C, D implement the allowed basis transformations within sets of N gauge fields of the same (Dirichlet or Neumann) BC type. The Eq. (A.1) then asserts that such $2N$ -dim system can always be factorized into N two-dimensional sub-systems (each is characterized by an $O(2)$ -rotation angle ϕ_i), whose KK decomposition is presented in Section 2.1. In addition, we specially note that (as it is evident from (A.3)–(A.6)) the basis rotations characterized by A, B, C, D do not have any effect on the set $\{\phi_1, \dots, \phi_N\}$. Those angles are the only relevant parameters encoded in (and unique to) the initial set of BCs.

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