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Entangled risks in incomplete FX markets

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ABSTRACT

We introduce the concept of risk entanglement in a preference-free setting to jointly explain the exchange rate volatility, cyclicality, and currency risk premia in the data. Risk entanglement specifies a subset of incomplete market models, in which nondiffusive or nonlog-normal shocks to exchange rates are not fully spanned by asset returns. When risks are entangled, there exist multiple pricing-consistent exchange rates, but none of them are equal to the ratio of the stochastic discount factors (SDFs) or their projections. Decoupling the exchange rate from the SDFs allows us to address key FX market patterns that are puzzling in international finance.

1. Introduction

A central research interest of international finance is to understand how the exchange rate is connected to asset pricing and macroeconomic fundamentals across countries. A key issue in traditional models is that in every, country aggregate consumption growth fully determines the stochastic discount factor (SDF) and the exchange rate is equal to the ratio of country-specific SDFs.[1] This implies a counterfactual relation between consumption and the exchange rate.

There are two complementary approaches to tackle this shortcoming. First, there is a literature that introduces a wedge between shocks to aggregate consumption growth and the SDF in each country. Then, the exchange rate can be disconnected from consumption while it is still equal to the ratio of country-specific SDFs. Examples are models of long-run risk (Colacito and Croce, 2011; Bansal and Shaliastovich, 2012; Colacito and Croce, 2013; Colacito et al., 2018a; 2018b), habit formation preferences (Verdelhan, 2010; Stathopoulos, 2017), disaster risks (Maggiori and Gabaix, 2015; Farhi and Gabaix, 2016; Farhi et al., 2015), frictions in shipping (Ready et al., 2017b; 2017a), trade network centrality (Richmond, 2019; Jiang and Richmond, 2020), government fiscal conditions (Jiang, 2021), heterogeneous agents (Kollmann, 2009; Fang, 2018), segmented asset markets (Chien et al., 2020), and financial intermediaries (Fang and Liu, 2020).

The second approach uses market incompleteness to introduce a wedge between the exchange rate and the ratio of SDFs (Zapatero, 1995; Sarkissian, 2003; Dumas et al., 2003; Chaieb and Errunza, 2007; Pavlova and Rigobon, 2007; Favilukis et al., 2015; Bakshi et al., 2018;

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[1] We refer to traditional models as consumption-based asset pricing models with a representative agent, constant relative risk aversion utility, independently and identically distributed shocks to aggregate consumption growth, and complete financial markets.

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Lustig and Verdelhan, 2019; Sandulescu et al., 2020). The SDF may still be fully determined by shocks to consumption growth in each country. Both approaches are important, and a model that combines both of them may be best suited to fit the data. However, in our analysis we want to distill the effect of market incompleteness for expositional purposes. Therefore, we follow Lustig and Verdelhan (2019) and assume that aggregate consumption growth fully determines the SDF in each country and focus on the second approach to investigate what type of market incompleteness is necessary and to what extent incompleteness can resolve international finance puzzles.

We make several important contributions. In contrast to the seminal paper by Lustig and Verdelhan (2019), we find that incomplete markets can fully reconcile the puzzles. We identify a novel market incompleteness characteristic, called risk entanglement, as the key feature. The analysis of Lustig and Verdelhan (2019) does not consider the risk entanglement settings in our paper. We explicitly model risks using a jump diffusion setting, which allows us to compute all quantities of interest exactly. This helps to gain insights into the economic tradeoffs and provides guidance for our calibration. In contrast, when risks are nonlog-normal, the setting of Lustig and Verdelhan (2019) does not offer an analytical expression for some quantities, as, for instance, the exchange rate cyclicality measure. Thus, our paper makes progress in this dimension and demonstrates that entangled jump risks (i.e., nonlog-normal risks) are necessary and able to reconcile international finance puzzles. Finally, our findings are useful to inform macro-finance models. We provide guidance about what type of incompleteness (i.e., risk entanglement) is desirable in a model to mitigate international finance puzzles.

Risk entanglement specifies a subset of incomplete market models in which nondiffusive or nonlog-normal shocks to the exchange rate are not fully spanned by traded assets. Risk entanglement generates a wedge between the exchange rate and the ratio of SDFs as well as the projected SDFs. The projected (or minimum variance) SDF of country I is the least squares projection of the SDF onto the asset return space denominated in currency I. We demonstrate that a wedge between the exchange rate and the ratio of projected SDFs can resolve international finance puzzles. When risks are entangled, there exist multiple pricing-consistent exchange rates, providing flexibility to obtain an exchange rate that explains empirical facts. In contrast, if markets are incomplete but risks are not entangled (i.e., if they are disentangled), then the exchange rate is still equal to the ratio of projected SDFs (albeit it generally differs from the ratio of SDFs), giving rise to the puzzles.

To illustrate this issue, consider the exchange rate cyclicality puzzle. When the ratio of country-specific consumption growth is regressed on the exchange rate growth, the slope coefficient is close to zero in the data (Backus and Smith, 1993). As mentioned previously, we follow Lustig and Verdelhan (2019) and assume that consumption growth fully determines the SDF in each country. As such, consumption growth and SDF are interchanged when confronting theoretical model predictions with data. In the case of the cyclicality puzzle, this means that the slope coefficient should be close to zero when we regress the ratio of SDFs on the exchange rate growth. Note that only the projected SDFs matter in this regression, while unspanned risks in the SDFs are irrelevant as they are orthogonal to the risks that affect the exchange rate. Thus, as long as risks are disentangled, the exchange rate is equal to the ratio of projected SDFs, the slope coefficient is one, and the cyclicality puzzle cannot be resolved. Similar issues arise with other international finance puzzles. Accordingly, we employ incomplete markets with entangled risks to resolve the puzzles as there is a wedge between the exchange rate and the ratio of projected SDFs.

Finally, we use numerical solutions to illustrate the quantitative ability of models with entangled risks to generate moments consistent with the data. Our calibration provides important insights to inform macro-finance models as to what type of market incompleteness and what kind of asset and risk configurations can mitigate international finance puzzles.

The key property of risk entanglement is that investors cannot perfectly hedge risks in the FX markets. Conversely, risks are disentangled if investors can perfectly hedge every risk that impacts the exchange rate. Note that a continuous time setting with only diffusion risks does not feature risk entanglement as the diffusion risk space can always be linearly transformed into an observationally equivalent space in which the number of diffusion risks that affect asset prices equals the number of nonredundant assets. Thus, if there are only diffusion risks, investors can perfectly hedge the shocks to the exchange rate.

It is important to note that entangled risks always imply incomplete markets, but the converse is not true. For instance, consider a setting where all risks that impact asset prices are spanned, while other risks, affecting asset return moments or SDFs but not asset prices, are unspanned. In this example, the markets are incomplete as some risks in the economy are unspanned, but risks are disentangled as all shocks to asset prices are spanned. Thus, entangled risks are a special case of market incompleteness. Whether risks are entangled or disentangled in incomplete markets depends upon how the risks are embedded in the asset markets. Finally, note that risks are always disentangled in complete markets.

Risk entanglement is a preference-free feature that arises in both integrated and segmented markets. Market segmentation allows for different valuations of the same asset by investors in different countries facing different restrictions in trading the asset. By construction, a segmented asset gives rise to a wedge between the exchange rate and projected SDFs. Risk entanglement generates a wedge with or without market segmentation. It can also perfectly replicate the wedge created by a large class of segmented markets considered in the literature in which segmented assets are present but not explicitly specified. This result highlights the modeling flexibility of the risk entanglement framework.

Our analysis focuses on the following three international finance puzzles. First, the slope coefficient is close to zero in a regression of the ratio of country-specific consumption growths on the exchange rate growth

(Backus and Smith, 1993; Lustig and Verdelhan, 2019). This contradicts the coefficient equal to one in traditional models that assume the exchange rate is equal to the ratio of country-specific SDFs and SDFs are fully determined by aggregate consumption growth. This is termed the exchange rate cyclicality puzzle. We show that entangled risks can reduce the slope coefficient close to zero in a regression of the ratio of country-specific SDFs on the exchange rate growth.

Second, Brandt et al. (2006) note that if the exchange rate is equal to the ratio of SDFs, then high equity premia (or volatile SDFs), together with a moderate volatility in the exchange rate growth, implies an almost perfect correlation between country-specific SDFs. If consumption growth fully determines the SDF in each country, then this further implies an almost perfect correlation between consumption growths across countries. This is at odds with the moderate cross-country correlations between consumption growths (or other macroeconomic quantities) in the data. This is the exchange rate volatility puzzle. We show that when risks are entangled, the exchange rate is not equal to the ratio of SDFs, and the exchange rate can be smooth while SDFs are volatile and low correlated.

Third, Fama (1984) finds that the uncovered interest parity (UIP) is violated in the data, and often high interest rate currencies appreciate against low interest rate currencies. This implies sizable currency premia (Lustig and Verdelhan, 2007). Traditional models are unable to explain the high currency premia. We show that a setting of entangled risks can generate currency risk premia that are equal to the average returns in the data.

Section 2 illustrates and formalizes the concept of risk entanglement. Section 3 demonstrates in a calibration that a model with entangled risks can match moments in the data. Section 4 concludes. Additional calibration results and technical details and proofs are in the Appendix.

2. Risk entanglement

This section first discusses the basic idea of risk entanglement and then illustrates it in a specific discrete setting (Section 2.1) before formalizing it in a general jump diffusion setting (Section 2.2). Intuitively, risk entanglement characterizes the configuration of risks in financial markets that some risks always appear together in financial assets and hence cannot be individually contracted in the markets. Risk entanglement is a special case of market incompleteness. Risk entanglement implies an incomplete market, but the reverse is not true.

In an international setting, when exchange rate risks are disentangled in asset markets, investors can perfectly hedge all FX market risks. Accordingly, combining an asset with an appropriate FX hedging portfolio yields the same risk bundle when denominated either in foreign or home currency. This implies that home and foreign investors trade the same risk bundles and the traded asset return spaces in the two currencies are identical. In this case, the exchange rate is mechanically the conversion factor (i.e., the ratio) of the projected SDFs in the two currencies.

In contrast, if risks are entangled, then investors cannot perfectly hedge FX market risks. Thus, the conversion of an asset from the home to the foreign currency changes the risk bundle, and this transformation cannot be reversed by combining the asset with a suitable hedging portfolio. As such, the return spaces of the same set of traded assets in integrated international financial markets are different when these returns are denominated in different currencies. As a result, the marginal pricings of individual exchange rate risks of home and foreign investors are not necessarily related to one another by the conversion factor of the exchange rate. Then, the exchange rate does not equal the ratio of projected SDFs. Note, however, that by no arbitrage, these investors still agree on the prices of traded assets when converted to some common currency.

2.1. Discrete settings

To illustrate risk entanglement in discrete settings, we consider a parsimonious model with two (home and foreign) countries $I \in \{H, F\}$, two dates $t \in \{0, 1\}$, and S possible future states $s \in \{1, \ldots, S\}$ at date $t = 1$. We assume that international financial markets are free of arbitrage opportunities, frictionless, integrated and the short-term risk-free bond B_I of every country $I \in \{H, F\}$ is a traded asset. Specifically, let the integrated international financial markets be spanned by a set $\{Y\}$ of $N+1$ basis assets. It is important to note that the basis assets in $\{Y\}$ can originate from either the home or the foreign economy. In an integrated market framework, they can also be financial derivatives traded in international markets. For an illustration, if $\{B_H, Y_{H1}, \ldots, Y_{HK}\}$ denotes the bond and K nonredundant traded assets that originate from the home economy, and $\{B_F, Y_{FK+2}, \ldots, Y_{FN}\}$ the bond and $N - K - 1$ other nonredundant traded assets from the foreign economy, then $\{Y\} = \{B_H, Y_{H1}, \ldots, Y_{HK}, \frac{B_F}{e}, \frac{Y_{FK+2}}{e}, \ldots, \frac{Y_{FN}}{e}\}$ is the set of returns denominated in the home currency of $N+1$ nonredundant internationally traded basis assets. Because markets are integrated, foreign investors trade the same $N+1$ financial basis assets but with returns converted to the foreign currency denomination $\{eY\} = \{eB_H, eY_{H1}, \ldots, eY_{HK}, B_F, Y_{FK+2}, \ldots, Y_{FN}\}$ by the exchange rate factor e. Hence, without loss of generality, we specify $N+1$ basis returns in the home currency denomination

$$\frac{B_H(s)}{B_{H0}} = 1 + r_H, \qquad \frac{Y_n(s)}{Y_{n0}} \qquad \forall s \in \{1, \ldots, S\}, n \in \{1, \ldots, N\}, \tag{1}$$

where B_H denotes the home bond, r_H the home interest rate, $Y_n(s)$ the payoff in state s, and Y_{n0} the current price of the n-th risky asset Y_n. Asset pricing in country $I \in \{H, F\}$ is characterized by the strictly positive SDF M_I associated with currency I. Hence, M_I prices risks from the perspective of country I.

Let the exchange rate denote the amount of the foreign currency that buys one unit of the home currency. Without loss of generality, the current $(t = 0)$ exchange rate is normalized to be one; $e_0 \equiv 1$. To determine the future $(t = 1)$ exchange rate $e(s)$, $s \in \{1, \ldots, S\}$, we take as given (i) the physical probability distribution $\{prob(s)\}$,

$s \in \{1, \ldots, S\}$, $\sum_{s=1}^{S} prob(s) = 1$, of future states; (ii) the growth of the home country SDF $\frac{M_H(s)}{M_{H0}}$; and of the foreign country $\frac{M_F(s)}{M_{F0}}$, and (iii) basis asset returns (1) specified in the home currency. The exchange rate determination starts with the foreign bond tradability assumption, or how the foreign bond as a traded asset is spanned by the given basis returns, $\frac{B_F(s)}{B_{F0}} = \frac{1+r_F}{e(s)} = \sum_{n=1}^{N} \alpha_n \frac{Y_n(s)}{Y_{n0}} + \left(1 - \sum_{n=1}^{N} \alpha_n\right)(1 + r_H)$ for all states s. This implies a portfolio representation of the exchange rate

$$e(s) = \frac{1 + r_F}{\sum_{n=1}^{N} \alpha_n \frac{Y_n(s)}{Y_{n0}} + (1 - \sum_{n=1}^{N} \alpha_n)(1 + r_H)}, \qquad s \in \{1, \ldots, S\}, \tag{2}$$

where $\alpha_n \in \mathbb{R}$ is the weight associated with risky asset Y_n. Employing this exchange rate in the pricing equations in the two currencies, and taking their difference, produces a balanced equation system of unknown weights $\{\alpha_n\}$

$$E_t\left[\left(\frac{M_F}{M_{F0}} e - \frac{M_H}{M_{H0}}\right)\frac{Y_n}{Y_{n0}}\right] = \sum_{s=1}^{S} prob(s)\left(\frac{M_F(s)}{M_{F0}} e(s) - \frac{M_H(s)}{M_{H0}}\right)\frac{Y_n(s)}{Y_{n0}} = 0, \qquad \forall n \in \{1, \ldots, N\}. \tag{3}$$

Note that this equation also holds for the home bond (with B_H replacing Y_n in (3)). We consider three specific asset market configurations to demonstrate the notion of risk entanglement. For simplicity, let the number of states be $S = 3$.

Complete markets

In the current discrete setting with three future states, markets are complete in the presence of any three nonredundant basis assets. Without loss of generality, let the basis assets be the three Arrow-Debreu (AD) securities $\{Y\} = \{A_s\}$, $s \in \{1, 2, 3\}$, that pay off (in units of the home currency) only in respective future states s: $A_s(s) = 1$ and $A_s(s') = 0$, $s' \neq s$ at $t = 1$. Let A_{s0} denote the price in the home currency of the AD security A_s at $t = 0$. We have $A_{s0} = prob(s)\frac{M_H(s)}{M_{H0}}$, $s \in \{1, 2, 3\}$. An application of system (3) for the AD securities yields state-by-state identities,

$$\frac{M_F(s)}{M_{F0}} e(s) - \frac{M_H(s)}{M_{H0}} = 0 \;\Rightarrow\; e(s) = \frac{\frac{M_H(s)}{M_{H0}}}{\frac{M_F(s)}{M_{F0}}}, \qquad \forall s \in \{1, 2, 3\}. \tag{4}$$

This is the well-known equality between the exchange rate and the ratio of country-specific SDFs in complete markets.

Incomplete markets: risk disentanglement

We now consider the first incomplete market setting with only two basis assets, the home bond and the first AD security, $\{Y\} = \{B_H, A_1\}$. In particular, the first AD security offers state-contingent payoffs $A_1(1) = 1$ and $A_1(s') = 0$, $s' \in \{2, 3\}$ at $t = 1$, and has the current price of A_{10} at $t = 0$ in units of the home currency. As a special case of (3), the Euler equations for the pricing of the first AD security in foreign and home currencies imply

$$E_t\left[\frac{M_F(s)}{M_{F0}} e(s) \frac{A_1(s)}{A_{10}}\right] = E_t\left[\frac{M_H(s)}{M_{H0}} \frac{A_1(s)}{A_{10}}\right]. \tag{5}$$

Since the first AD security has a payoff of 1 in state $s = 1$ and 0 otherwise, the above equality implies

$$\frac{M_H(1)}{M_{H0}} prob(1) = \frac{M_F(1)}{M_{F0}} e(1) prob(1) \;\Rightarrow\; e(1) = \frac{\frac{M_H(1)}{M_{H0}}}{\frac{M_F(1)}{M_{F0}}}. \tag{6}$$

Intuitively, since the first state is singly contracted by the presence of the first AD security A_1, the exchange rate in the first state $s = 1$ is fully determined by SDFs $M_H(1)$, $M_F(1)$ in this state. The Euler Eq. (3) for the pricing of the home bond in the home and foreign currencies becomes

$$\frac{M_H(1)}{M_{H0}} prob(1) + \sum_{s=2}^{3} \frac{M_H(s)}{M_{H0}} prob(s) = \frac{M_F(1)}{M_{F0}} e(1) prob(1) + \sum_{s=2}^{3} \frac{M_F(s)}{M_{F0}} e(s) prob(s). \tag{7}$$

The first term cancels on both sides of the above equation due to Eq. (6), and we obtain

$$\sum_{s=2}^{3} \frac{M_H(s)}{M_{H0}} prob(s) = \sum_{s=2}^{3} \frac{M_F(s)}{M_{F0}} e(s) prob(s). \tag{8}$$

Because the AD security A_1 has an identical payoff of zero in states $s \in \{2, 3\}$ and it is the only risky basis asset in the current setting, the portfolio representation (2) implies that the exchange rate in these states is identical. The above equality then produces

$$e(2) = e(3) = \frac{\frac{M_H(2)}{M_{H0}} prob(2) + \frac{M_H(3)}{M_{H0}} prob(3)}{\frac{M_F(2)}{M_{F0}} prob(2) + \frac{M_F(3)}{M_{F0}} prob(3)}. \tag{9}$$

This equation together with (6) presents a unique exchange rate solution in the current incomplete market setting. The exchange rate solution equals the ratio of the home and foreign projected SDFs on the traded asset return space as in Brandt et al. (2006). Intuitively, this uniqueness arises from the fact that basis assets $\{B_H, A_1\}$ offer identical payoffs in states $\{2, 3\}$; hence the financial markets spanned by basis assets do not distinguish these two states. There are effectively only two exchange rate states; $s = 1$ and $s \in \{2, 3\}$, and two basis assets $\{B_H, A_1\}$ suffice to completely hedge these two exchange rate states. As a result, the current market setting features risk disentanglement and a unique exchange rate. However, note that the economic distinction between the states $s \in \{2, 3\}$ is still relevant because they are distinguished by SDFs; $M_H(2) \neq M_H(3)$ or $M_F(2) \neq M_F(3)$. This distinction in SDFs may be due to various reasons, e.g., shocks to the marginal investor that are outside of international financial markets or shocks to moments of traded asset returns.

Incomplete markets: risk entanglement

We now consider a second incomplete market setting also with two basis assets, the home bond and a risky asset, $\{Y\} = \{B_H, Y\}$. At $t = 1$, let Y's state-contingent payoffs be $\{Y(s)\} = \{Y(1), Y(2), 0\}$ in states $s \in \{1, 2, 3\}$, such that $Y(1) \neq Y(2)$. At $t = 0$, the risky asset Y has current price of

$$Y_0 = \sum_{s=1}^{3} prob(s)\frac{M_H(s)}{M_{H0}}Y(s) = prob(1)\frac{M_H(1)}{M_{H0}}Y(1) + prob(2)\frac{M_H(2)}{M_{H0}}Y(2) \quad (10)$$

in units of the home currency. The portfolio representation (2) then presents the exchange rate in terms of portfolio weight α for every state,

$$e(1) = \frac{1+r_F}{\alpha\frac{Y(1)}{Y_0} + (1-\alpha)(1+r_H)}, \quad e(2) = \frac{1+r_F}{\alpha\frac{Y(2)}{Y_0} + (1-\alpha)(1+r_H)}, \quad e(3) = \frac{1+r_F}{(1-\alpha)(1+r_H)}. \quad (11)$$

The substitution of this exchange rate into pricing Eq. (3) of the risky asset Y then produces a nonlinear (quadratic) equation of portfolio weight α. In fact, with the substitution of the exchange rate (11), the pricing equation of asset Y in the foreign currency, $E_0\left[\frac{M_F}{M_{F0}}e\frac{Y}{Y_0}\right] = 1$, is a quadratic equation in α,

$$prob(1)\frac{M_F(1)}{M_{F0}}\frac{1+r_F}{\alpha\frac{Y(1)}{Y_0} + (1-\alpha)(1+r_H)}\frac{Y(1)}{Y_0} + prob(2)\frac{M_F(2)}{M_{F0}}\frac{1+r_F}{\alpha\frac{Y(2)}{Y_0} + (1-\alpha)(1+r_H)}\frac{Y(2)}{Y_0} = 1. \quad (12)$$

There exist multiple exchange rate solutions consistent with the setup's inputs $\{M_H, M_F, \{B_H, Y\}, prob(s)\}$. Each exchange rate solution has different values across the three states, so these states are three genuinely distinct exchange rate states. Further, none of the AD securities $\{A_1, A_2, A_3\}$ can be constructed as a portfolio of the basis assets $\{B_H, Y\}$. Hence none of the three distinct exchange rate states $s \in \{1, 2, 3\}$ can be individually contracted in financial markets. This incomplete market setting therefore features risk entanglement (formalized in Definition 1 below) and multiple exchange rate solutions.

We observe that incomplete markets alone do not necessarily give rise to multiple exchange rate solutions as illustrated in Eq. (9). It is risk entanglement that enables this multiplicity. We note that multiple exchange rate solutions in risk entanglement settings are not multiple equilibria of the exchange rate. Instead, each of the exchange rate solutions points to a distinct foreign bond specification B_F in terms of the given basis assets. When we start out specifying a particular foreign bond B_F (instead of specifying basis assets $\{Y\}$), we find a single exchange rate solution that is consistent with this particular foreign bond specification.

2.2. Risk entanglement: general case

We now formalize the concept of risk entanglement and its effects on the exchange rate determination in a general setting of continuous time and state space. Apart from the no-arbitrage pricing restriction, we state two assumptions.

Assumption 1. International asset markets are fully integrated.

Assumption 2. Short-term risk-free bonds of every country are traded assets.

We consider the following generic specification of country I's SDF, $I \in \{H, F\}$

$$\frac{M_{It+dt}}{M_{It}} = 1 - r_{It}dt - \eta_{It}'dZ_t + \sum_{i\in\mathcal{J}_I}\left(e^{\Delta_{Iit}} - 1\right)(d\mathcal{N}_{it} - \lambda_{it}dt), \quad (13)$$

with $M_{I0} = 1, \quad t \in [0, \infty), \quad I \in \{H, F\}, \quad d\mathcal{N}_{it} \in \text{Poisson}(\lambda_{it}),$

where $r_{It} \in \mathbb{R}$ is country I's instantaneously risk-free rate, vector $Z_t \in \mathbb{R}^d$ denotes d independent standard Brownian motions (i.e., diffusion risks), $\eta_{It} \in \mathbb{R}^d$ the prices of the associated risks in currency I, and $'$ the matrix transpose. Jump of type i is characterized by an independent Poisson counting process $\mathcal{N}_{it}$ of arrival intensity λ_{it}. $\mathcal{J}_I$ denotes the set of jump types i that induce nonzero jump sizes $\Delta_{Iit} \in \mathbb{R}\setminus\{0\}$ in SDF M_I. Hence $\mathcal{J}_I$ is also the set of jump risks that are priced in currency I.

Let $\{Y\}$ be a set of nonredundant internationally traded basis assets. Any other traded asset is a portfolio of the basis assets in $\{Y\}$. While the basis assets in $\{Y\}$ can originate from either the home or the foreign economy, we assume that their returns are given in the home currency denomination without loss of generality as elaborated in the discrete setting,

$$\frac{B_{Ht+dt}}{B_{Ht}} = 1 + r_H dt, \qquad \frac{Y_{t+dt}}{Y_t} = 1 + \mu_{Yt}dt + \sigma_{Yt}'dZ_t + \sum_{i\in\mathcal{J}_Y}\left(e^{\Delta_{Yit}} - 1\right)(d\mathcal{N}_{it} - \lambda_{it}dt), \quad (14)$$

where B_H denotes the home bond; Y a generic risky asset among $\{Y_1, \dots, Y_N\}$; $\mathcal{J}_Y$ the set of jump types that impact asset Y's return; $\mu_{Yt} \in \mathbb{R}$ and $\sigma_{Yt} \in \mathbb{R}^d$ asset Y's expected return and diffusion volatility, respectively; and $\Delta_{Yit} \in \mathbb{R}\setminus\{0\}$ the jump size of asset Y's return associated with jump type i.[2] The markets are possibly incomplete as jump diffusion risks impacting either asset payoffs or SDFs are not perfectly hedged by holding portfolios of traded assets.

To determine the exchange rate, we first express it as a generic jump diffusion process

$$\frac{e_{t+dt}}{e_t} = 1 + \mu_e dt + \sigma_e' dZ_t + \sum_{i\in\mathcal{J}_{\{Y\}}}\left(e^{\Delta_{ei}} - 1\right)d\mathcal{N}_{it} \quad (15)$$

and then solve for the mean μ_e, volatility σ_e, and jump sizes $\{\Delta_{ei}\}$. Because the foreign bond is a traded asset (Assumptions 2), its return denominated in the home currency is spanned by the basis returns (14). Therefore, in this market-based approach, jump types that impact the

[2] In general, $r_{It}, \eta_{It}, \Delta_{Iit}, \lambda_{it}, \mu_{Yt}, \sigma_{Yt}$, and Δ_{Yit} can be stochastic processes adapted to the information structure generated by Z_t and $\{\mathcal{N}_{it}\}$, $\forall i$. We drop the time index whenever such an omission does not create ambiguities.

exchange rate are from the set $\mathcal{J}_{\{Y\}}$ of jump types that impact basis asset returns $\{Y\}$. This spanning produces the following portfolio representation for the exchange rate, generalizing (2)

$$\frac{e_t}{e_{t+dt}} = \frac{1}{1+r_F dt}\left\{\left(1-\sum_{n=1}^{N}\alpha_n\right)\frac{B_{H,t+dt}}{B_{H,t}} + \sum_{n=1}^{N}\alpha_n \frac{Y_{nt+dt}}{Y_{nt}}\right\}. \tag{16}$$

The substitution of this representation and the exchange rate process (15) into the N Euler pricing equations for the N risky basis assets in the foreign currency denomination, $E_t[\frac{M_{Ft+dt}}{M_{Ft}}\frac{Y_{t+dt}}{Y_t}] = 1$, $Y \in \{Y_1, \ldots, Y_N\}$, generates a balanced system of N equations and N unknown weights $\{\alpha_n\}$ (see Appendix B).[3]

$$\sigma_Y'(\eta_H - \eta_F + \sigma_e) + \sum_{i\in(\mathcal{J}_Y\cap\mathcal{J}_F)}\lambda_i e^{\Delta_{Fi}+\Delta_{ei}}\left(e^{\Delta_{Yi}}-1\right) + \sum_{i\in(\mathcal{J}_Y\setminus\mathcal{J}_F)}\lambda_i e^{\Delta_{ei}}\left(e^{\Delta_{Yi}}-1\right)$$
$$= \sum_{i\in(\mathcal{J}_Y\cap\mathcal{J}_H)}\lambda_i\left(e^{\Delta_{Hi}}-1\right)\left(e^{\Delta_{Yi}}-1\right) + \sum_{i\in\mathcal{J}_Y}\lambda_i\left(e^{\Delta_{Yi}}-1\right), \qquad \forall Y \in \{Y_1,\ldots,Y_N\}, \tag{17}$$

where $\mathcal{J}_A \cap \mathcal{J}_B$ denotes the set of jump types that are common to both sets $\mathcal{J}_A$ and $\mathcal{J}_B$, and $\mathcal{J}_A \setminus \mathcal{J}_B$ the set of jump types that belongs to $\mathcal{J}_A$ but not $\mathcal{J}_B$. As a result of the portfolio representation (16), the moments μ_e, σ_e, $\{\Delta_{ei}\}$ of the exchange rate are functions of the portfolio weights $\{\alpha_n\}$. Therefore, (17) is a balanced system of (nonlinear) equations of the N unknowns $\{\alpha_n\}$. The jump set notation shows that unhedged jump risks, that is, those impacting at least one SDF but not asset returns $i \in (\mathcal{J}_H \setminus \mathcal{J}_Y)$ and $i \in (\mathcal{J}_F \setminus \mathcal{J}_Y)$, are decoupled from the above equation system. Hence, these risks do not matter for the exchange rate determination from the market perspective (Protocol 1). Similarly, unhedged diffusion risks, i.e. components of η_H or η_F that are uncorrelated with the diffusion σ_Y of every asset return Y, do not contribute to the exchange rate determination. On the other hand, idiosyncratic jump and diffusion risks, those impacting asset returns but not SDFs, remain relevant to system (17) and the exchange rate solution. Before further analyzing this system, we summarize the key steps to determine the exchange rate in the current general setting.

Protocol 1 (No-arbitrage determination of the exchange rate). Step 1: We take as given (i) home and foreign SDFs M_{Ht}, M_{Ft} (13), and (ii) basis asset returns $\left\{\frac{B_{Ht+dt}}{B_{Ht}}, \frac{Y_{nt+dt}}{Y_{nt}}\right\}$, $n \in \{1,\ldots,N\}$ (14) specified (without loss of generality) from the perspective of home investors.

Step 2: We construct a portfolio representation (16) for the exchange rate based on the foreign bond tradability assumption. The pricing of basis assets in the foreign currency by M_F forms an equation system (17) to solve for all possible portfolio weights and exchange rates in this representation.

Step 3: Ex-post, each specific exchange rate solution e_t identifies a distinct specification of asset returns in the foreign currency. If we begin with a particular specification ex-ante, then we obtain a unique exchange rate e_t that identifies the particular specification under consideration.

This protocol belongs to the literature of market-based determination of the exchange rate. Our innovations are incorporated in the construction of the exchange rate portfolio representation (Step 2) and the ex-post identification of the market model that uniquely gives rise to a specific exchange rate solution (Step 3). In particular, the exchange rate portfolio representation allows us to quantify and obtain all possible pricing-consistent exchange rates as solutions of the equation system (17). This is a balanced system of nonlinear equations of exchange rate weights $\{\alpha_n\}$ (16). Because risk entanglement in international asset markets determines the non linearity of this equation system, risk entanglement is important to both the quantitative and qualitative properties of the exchange rate solutions. We elaborate on the key role of entangled risks in the exchange rate determination, starting with a formal definition of risk entanglement.

Definition 1 (Risk entanglement). Risks are entangled in asset markets if there exist jump risks impacting instantaneous returns of some traded asset Y_t that cannot be perfectly hedged by any portfolio P_t of traded assets $\{Y\}$. Otherwise, risks are disentangled in asset markets.

Mathematically, risks are entangled in asset markets of $N+1$ basis assets $\{Y\} \equiv \{B_H, Y_1, \ldots, Y_N\}$ if $\exists \mathcal{N}_i : \Delta_{Y_n i} \neq 0$ for $Y_n \in \{Y\}$ and $\nexists \alpha \in \mathbb{R}^N : \frac{P_{t+dt}}{P_t} \equiv \left(1 - \sum_{n=1}^{N}\alpha_n\right)\frac{B_{Ht+dt}}{B_{Ht}} + \sum_{n=1}^{N}\alpha_n\frac{Y_{nt+dt}}{Y_{nt}} = 1 + \mu_P dt + \left(e^{\Delta_{Pi}}-1\right)(d\mathcal{N}_{it} - \lambda_i dt)$ and $\Delta_{Pi} \neq 0$.

Note that risks to instantaneous returns $\frac{Y_{nt+dt}}{Y_{nt}}$ are also the risks to asset prices Y_{nt} and are central to the definition of risk entanglement. Several observations concerning risk entanglement are in order.

First, when risk-free bonds are traded, exchange rate risks are the risks that home investors face when trading the foreign bond. In this case, exchange rate risks are a part of asset market risks and, as such, are covered by the above definition. Second, while our setup assumes market integration (Assumption 1), risk entanglement and its effect on the exchange rate determination arise in both integrated and segmented market settings. In particular, the current integrated market setting can replicate the asset pricing of segmented markets in which only the home and foreign risk-free bonds are explicit traded assets. Finally, we observe that markets that are impacted only by diffusion risks always feature risk disentanglement. To illustrate, suppose d independent diffusion risks impact $N < d$ non redundant asset returns. We can always redefine (i.e., linearly combine) and partition the d original diffusion risks into two orthogonal subsets; N diffusion risks are disentangled by N asset returns, and the remaining $(d-N)$ diffusion risks drop out because they are uncorrelated with, and not detected by, asset returns. This asset market setting hence features risk disentanglement per Definition 1. Thus, in continuous settings, risk entanglement requires jump risks. Without jumps or in the limit of jump sizes approaching zero, the non linearity of the exchange rate determination Eq. (17) disappears, yielding a

[3] Note that the Euler pricing equation for the risk-free bond B_H in the foreign currency is redundant as it is implied from the N Euler pricing equations for the N risky assets and the portfolio representation (16).

single exchange rate solution. We summarize this observation in the following remark.

Remark 1. Risks are always disentangled in every (incomplete) asset market that features only diffusion risks.

Intuitively, in the absence of risk entanglement, for every (diffusion or jump) risk i that impacts the exchange rate (15), there is an asset Y_i that loads only on that risk. The Euler equation that prices this asset Y_i then concerns only one exchange rate risk. As a result, the equation system (17) is completely decoupled into individual equations, each pertains to and solves for the unique respective exchange rate weight α_i. In contrast, when risks are entangled in asset markets, the equation system (17) remains coupled and non linear in the exchange rate weights $\{\alpha_i\}$. As a result, multiple exchange rate solutions exist. Theorem 1 formalizes and strengthens this intuitive observation, whose proof is relegated to Appendix B.

Theorem 1. Assuming market integration and bond tradability (Assumptions 1 and 2), the equation system determining the exchange rate is non linear, and hence there exist multiple exchange rate solutions, if and only if the risks are entangled in asset markets.

In financial markets with multiple risks entangled in few traded assets, price data (observed in all currencies) is insufficient to unambiguously infer the exchange rate in every individual risk state. This gives rise to multiple pricing-consistent exchange rates. Exchange rate solutions depend not only on the given SDFs but also on the specificity in which risks are contracted in traded asset returns, that is, risk entanglement. This theorem suggests the use of risk entanglement to widen the set of pricing-consistent exchange rates and to enrich their associated dynamics as a possible way to better fit pricing and macro dynamics. When a fit is achieved, the specific market model that delivers this match ex-post is also identified (Step 3, Protocol 1).

It is known in the literature that the exchange rate is unique in complete markets (Saa-Requejo, 1994) and incomplete markets with pure-diffusion risks (Brandt et al., 2006). Several international finance puzzles also arise in these two settings. We observe that both settings belong to the class of risk disentanglement (Remark 1). Theorem 1 not only confirms the uniqueness of the exchange rate in the two settings but also identifies a deviation from these settings, or risk entanglement, as a potential approach to decouple the exchange rate from the ratio of projected SDFs and to reconcile international finance puzzles.

3. Calibration

We provide a calibration to demonstrate the quantitative capability of a model with entangled risks. Our model can generate (i) a smooth exchange rate while SDFs are volatile and low correlated, (ii) a low correlation between the exchange rate and the ratio of SDFs, and (iii) a currency premium comparable to average FX returns in the data. We further analyze and compare various market settings to dissect the quantitative effects of risk entanglement. Our calibration shows that not only the magnitude of entangled risks but also the differentials in their prices across currency denominations are important for the dynamics of the exchange rate solutions.

3.1. Empirical moments

Before we present our calibrated model, we start with a description of 28 empirical moments that we aim to match. We analyze the returns of a global stock market (MKT), a global equity high-minus-low book-to-market portfolio (HML^{GE}), and a currency carry trade (CT). We use data from July 1990 to February 2016. Monthly returns for the MKT and HML^{GE} are from Kenneth French's website.[4]

The CT is constructed from 28 spot and forward exchange rates against the USD provided by Barclays Bank International and Reuters via Datastream.[5] Monthly returns of CT are given by the pair trade that borrows in the currency with the lowest and invests in the currency with the highest one-month interest rate. We are interested in a pair trade (rather than a diversified currency portfolio) as there are only two currencies in our model. We focus on the trade between the highest and lowest interest rate currencies (rather than an average over all currencies) as the mean return is higher, and thus it is more challenging to explain the currency premium and reconcile the international finance puzzles.

We further use quarterly consumption growth data from the national accounts provided by Organisation for Economic Co-operation and Development (OECD).[6] At the end of every quarter, we choose the country with the highest (lowest) interest rate and use the consumption growth over the subsequent quarter to construct a consumption growth time series for the country with the highest (lowest) interest rate and denote it as Δc_{high} (Δc_{low}). Finally, we construct quarterly CT returns that we can compare to the quarterly consumption growth time series.

Table 1 reports estimated moments and 90% confidence intervals, which are reported in square brackets below the point estimates, for MKT, HML^{GE}, and CT. We use 10,000 bootstrap samples to construct the confidence intervals.

The average annualized interest rate differential between the highest and lowest interest rate currencies is 14% in our sample. In our model, returns are denominated in the home currency. We adopt the convention that the home country is the economy with the highest interest rate.[7] Hence, in our estimation, every month we convert returns into the currency that is currently associated with the highest interest rate. This allows us to compare the

[4] https://mba.tuck.dartmouth.edu/pages/faculty/ken.french/Data_Library/f-f_3developed.html.

[5] Countries: Australia, Belgium, Brazil, Canada, Czech Republic, Denmark, France, Germany, Greece, Hungary, Iceland, Ireland, Italy, Japan, Mexico, Netherlands, New Zealand, Norway, Poland, Portugal, Singapore, South Africa, South Korea, Spain, Sweden, Switzerland, Taiwan, and the United Kingdom. The euro was introduced in January 1999 and replaces the currencies of countries which have joined the euro after that date.

[6] https://stats.oecd.org/.

[7] Our calibration analysis also presents the pricing results from the perspective of a low interest rate currency. Thus, our convention is without loss of generality.

Table 1
Empirical moments.

	MKT	HML^{GE}	CT
$r_{high} - r_{low}$			0.140 [0.136,0.145]
Mean (r_{high} currency)	0.042 [-0.009,0.092]	0.039 [0.013,0.064]	0.048 [-0.007,0.100]
Mean (r_{low} currency)			0.073 [0.022,0.122]
Volatility	0.155 [0.140,0.171]	0.081 [0.070,0.091]	0.167 [0.141,0.194]
Skewness	-0.970 [-1.447,-0.417]	0.566 [-0.391,1.396]	-2.105 [-2.823,-0.826]
Kurtosis	5.812 [3.489,7.954]	8.467 [5.874,10.228]	13.149 [4.691,17.814]
Entropy	0.012 [0.010,0.015]	0.003 [0.002,0.004]	0.015 [0.010,0.021]
Corr(MKT,y)		-0.175 [-0.271,-0.071]	0.243 [0.050,0.401]
Corr(HML^{GE}, y)			-0.030 [-0.119,0.062]
Co-skew(MKT,MKT,y)		0.122 [-0.040,0.293]	-0.865 [-1.423,-0.091]
Co-skew(HML^{GE},HML^{GE}, y)	-0.284 [-0.590,0.045]		0.212 [0.017,0.388]
Co-skew(MKT,y, y)			-1.092 [-1.740,-0.003]
Co-skew(HML^{GE}, y, y)			0.196 [-0.073,0.462]
Corr($\frac{\Delta c_{high}}{\Delta c_{low}}$, y)			-0.001 [-0.132,0.133]
β_{BS}			-0.000 [-0.027,0.024]
Corr(Δc_{high}, Δc_{low})			0.182 [-0.017,0.363]

Notes: The Table 1 reports annualized moments and 90% confidence intervals. Confidence intervals are constructed using 10,000 bootstrap samples. MKT is the monthly excess return of the global value-weighted stock market and HML^{GE} the global high-minus-low book-to-market sorted stock portfolio. MKT and HML^{GE} returns are from Kenneth French's website. CT are monthly carry trade returns of borrowing in the currency with the lowest and lending in the one with the highest interest rate. CT returns are computed using spot and 1-month forward exchange rates of 28 currencies against the USD provided by Barclays Bank International and Reuters via Datastream. Δc_{high} (Δc_{low}) is quarter t consumption growth in the currency with the highest (lowest) interest rate in that quarter. Consumption data are from the OECD. $r_{high} - r_{low}$ is the average interest rate differential between the highest and lowest interest rate currencies. In the row labeled "Mean (r_{low} currency)" we convert the CT returns in month t into the currency with the lowest interest rate in month t to estimate the average carry trade return denominated in the low interest rate currency. To estimate all other moments we convert month t returns in the currency with the highest interest rate in month t. Variable y in the correlation and co-skewness terms refers to the corresponding variables in the column heads. The data are from July 1990 to February 2016.

returns in the data to the model-implied returns denominated in the home currency. The average annualized excess returns of the MKT, HML^{GE}, and CT are 4.2%, 3.9%, and 4.8% when denominated in the high interest rate currency. Moreover, the average annualized excess return of CT is 7.3% when converted into the low interest rate currency. The considerable difference of 2.5% in the CT premium across the high and low interest rate currencies is consistent with other estimates in the literature (Maurer et al., 2019). Note that the MKT and HML^{GE} converted into the low interest rate currency denomination are a combination of a currency trade and the investment in MKT or HML^{GE} in the high interest rate currency. Hence, these moments do not provide new information, and we omit them in the calibration.

The volatility of CT is 16.7%, which is higher than the value of 11.21% reported by Lustig and Verdelhan (2019). While their paper reports the average exchange rate volatility across all exchange rates in a set of 15 developed currencies, we estimate the volatility of the exchange rate between the lowest and highest interest rate currencies in a larger set of 29 developed and emerging market currencies. In turn, our currency premium is higher than what they report, that is, 6.4% versus 4.8% when denominated in USD.

The MKT and CT have negative skewness, which captures the occasional crashes in stock and currency markets. In contrast, the HML^{GE} has a skewness close to zero. All correlations are close to zero. The co-skewness between the HML^{GE} and the MKT or CT are close to zero, while the co-skewness between the MKT and CT is negative.

The correlation between $\frac{\Delta c_{high}}{\Delta c_{low}}$ and the exchange rate growth (denoted as Corr($\frac{\Delta c_{high}}{\Delta c_{low}}$, y) in the table) as well as the slope coefficient β_{BS} in the regression of $\frac{\Delta c_{high}}{\Delta c_{low}}$ on the exchange rate growth are both zero. These estimates are consistent with the findings of Backus and Smith (1993) and the values reported by Lustig and Verdelhan (2019). Finally, the correlation between Δc_{high} and Δc_{low} is 0.182 and not statistically different from zero, which confirms the estimates provided by Brandt et al. (2006) and Lustig and Verdelhan (2019).

3.2. Entangled versus disentangled jump risks

Our setup follows Protocol 1. We specify the SDFs and basis asset returns denominated in the home currency and then solve system (17), which pins down the pricing-consistent exchange rates that satisfy the portfolio representation (16). Our calibration features $d = 3$ diffusion risks $dZ_{1t}, dZ_{2t}, dZ_{3t}$, and $j = 2$ Poisson jump risks $d\mathcal{N}_{1t}, d\mathcal{N}_{2t}$. In addition to the risk-free bond B_H, we consider as the basis assets denominated in the home currency a subset of the following five nonredundant risky assets. Y_{MKT} represents the global stock market, $Y_{HML^{GE}}$ captures the global high-minus-low book-to-market portfolio, Y_3 only loads on jump risk $d\mathcal{N}_{2t}$, Y_4 only loads on jump risk $d\mathcal{N}_{1t}$, and Y_5 only loads on diffusion risk dZ_{1t}. Markets are complete if all five risky basis assets are admissible and are incomplete otherwise. Jump risk $d\mathcal{N}_{2t}$ ($d\mathcal{N}_{1t}$) is disentangled from all other risks if and only if Y_3 (Y_4) is admissible. Note that Y_3 (Y_4) disentangles $d\mathcal{N}_{2t}$ ($d\mathcal{N}_{1t}$) irrespective of the size of the jump risk loading Δ_{Y_3} (Δ_{Y_4}), provided it is different from zero. Thus, the magnitudes of Δ_{Y_3} and Δ_{Y_4} do not affect the results in our analysis, and they can be chosen freely in $\mathbb{R} \setminus \{0\}$. Similarly, σ_{Y_5} can be chosen freely in $\mathbb{R} \setminus \{0\}$.

Our reduced-form model takes SDFs as given, and therefore interest rates (which are the drifts of the SDF processes) are exogenous. Since our convention is that the interest rate is higher in the home than in the foreign country, we set the interest rate differential to be $r_H - r_F =$ 14% to match the data. The carry trade strategy borrows e_t units of the foreign currency and lends one unit of the home currency. The excess return is $CT = (1 + r_H dt) - (1 + r_F dt)\frac{e_t}{e_{t+dt}}$ (see Eq. (B.17) for details). Finally, note that the choice of the interest rates in the home and foreign currencies have no implications on the pricing-consistent exchange rates or any of the asset pricing moments that we aim to match. Thus, our model does not impose any restrictions on the interest rate processes, and interest rates can be freely chosen in $\mathbb{R}$. This reduced-form model feature is the same as in Lustig and Verdelhan (2019), and it is in contrast to structural models that posit a general equilibrium relation between interest rates and the risk loadings of the SDFs.

The aim of our calibration is to match 28 moments in Table 1 (i.e., all moments except for $r_{high} - r_{low}$). In addition, we impose a symmetry condition that home and foreign SDFs have the same volatility. This condition is interesting as models in the literature often use an asymmetry to generate a high FX market premium. However, we show that such an asymmetry is not necessary in our calibration.

We need to choose values for the two jump intensity parameters λ_1, λ_2, the five risk loadings of each SDF M_H and M_F (i.e., $-\eta_{Hd1}, -\eta_{Fd}$ $\forall d \in \{1, 2, 3\}$, $\Delta_{Hj}, \Delta_{Fj} \forall j \in \{1, 2\}$), and the five risk loadings of each risky basis asset Y_{MKT} and $Y_{HML^{GE}}$ (i.e., $\sigma_{Y_{MKT}d}, \sigma_{Y_{HML^{GE}}d}$ $\forall d \in \{1, 2, 3\}$, $\Delta_{Y_{MKT}j}, \Delta_{Y_{HML^{GE}}j}$ $\forall j \in \{1, 2\}$). In total these are values for 22 parameters. Accordingly, our calibration is overidentified with 29 conditions and 22 parameters. It is important to note that even if the model was exactly or underidentified, it is not necessarily true that the model can fit the moments and resolve the international finance puzzles. For instance, if markets are complete, then it is well known that several puzzles persist, and we cannot fit at least some moments, even if we have more parameters to choose than moments to fit.

An advantage of our model and the approach to specify the individual diffusion and jump risk processes is that we can compute all return moments in closed form (see the formulas in Appendix B.2). This allows us to efficiently search the parameter space for 22 values that generate model-implied moments to fit the empirical counterparts. In contrast, this is not possible in nonlog-normal settings in the model of Lustig and Verdelhan (2019) as they do not explicitly model the individual risk sources and closed-form solutions are not available for several moments that we consider.

The top panel in Table 2 specifies the jump intensities, the SDFs, and the risky basis assets denominated in the home currency. The jump intensities are $\lambda_1 = 0.115$ and $\lambda_2 = 0.126$. Thus, each jump is expected to hit the economy about once every eight or nine years.

The home and foreign SDFs M_{Ht} and M_{Ft} load on all five risks. As such, all risks are priced. Consistent with the bound of Hansen and Jagannathan (1991), the SDFs are volatile in order to generate high equity and FX premia. The volatilities of the home and foreign SDFs are both 86.7%, which we report on the second last row in the bottom panel of Table 2. The volatility accounts for diffusion and jump risks. As aforementioned, we impose the symmetry condition that the home and foreign SDFs have the same volatility.

Jump risks are important. The risk loadings of the home and foreign SDFs on jump $d\mathcal{N}_{1t}$ are $\Delta_{H1} = 0.6501$ and $\Delta_{F1} = -1.0785$. The corresponding loadings on jump $d\mathcal{N}_{2t}$ are $\Delta_{H2} = 0.5954$ and $\Delta_{F2} = 0.9590$. We show in the sensitivity analysis in Appendix A.1 as to how changes in these jump risk loadings affect the exchange rate. Moreover, in Appendix A.2, we provide an alternative calibration where we assume that both the home and foreign SDFs always have positive jump risk exposures, that is, jumps are adverse events to investors in both countries.

The correlation between the two SDFs is 18.2%, which we report on the last row in the bottom panel of Table 2. The correlation accounts for diffusion and jump risks. This is exactly equal to the point estimate of the correlation between the consumption growth of the high and low interest rate countries in the data, which we report on the last row in Table 1. We follow Backus and Smith (1993), Brandt et al. (2006), and Lustig and Verdelhan (2019) and replace consumption growth by SDFs when we aim to match moments. The implicit assumption is that SDFs are decreasing in consumption growth. As mentioned in the introduction, there is a large body of literature that breaks the tight link between country-specific SDFs and local aggregate consumption growth. In these models SDFs are typically highly correlated across countries, while the cross-country correlation between consumption growth is close to zero. Our paper is complementary. We aim to show that it is possible to reconcile FX market puzzles when risks are entangled even if the cross-country correlation of SDFs is close to zero. Structural features that allow for a higher correlation between SDFs combined with entangled risks may fit the data even better.

For Y_{MKT} and $Y_{HML^{GE}}$, we choose risk loadings such that they fit the first four moments, the entropy, the correlation, and co-skewness of the global MKT and HML^{GE} portfolios in the data. The model-implied moments are in the first two columns in the bottom panel of Table 2. The model matches 12 out of 13 moments, except the kurtosis of $Y_{HML^{GE}}$, which is too small in the model. We conclude that Y_{MKT} and $Y_{HML^{GE}}$ are accurate representations of the MKT and HML^{GE} in the data.

The middle panel in Table 2 presents the risk loadings of the pricing-consistent exchange rates for four calibrations. Recall that when risks are entangled, there exist multiple pricing-consistent exchange rates. In such a case, we report the exchange rate that most closely matches the empirical moments, that is, the exchange rate that generates the smallest sum of absolute deviations between model-implied and empirical moments. Exchange rate e_I corresponds to market setting (I) in which Y_{MKT}, $Y_{HML^{GE}}$, and Y_3 are admissible risky basis assets. Thus, markets are incomplete and jump risk $d\mathcal{N}_{1t}$ is entangled, while $d\mathcal{N}_{2t}$ is disentangled. In market setting II (with exchange rate e_{II}), Y_{MKT}, Y_3, and Y_4 are admissible risky basis assets. In market setting (III), Y_{MKT}, $Y_{HML^{GE}}$, Y_3, and Y_4 are admissible risky basis assets. Markets are incomplete but both jumps are disentangled in settings (II) and (III). Finally, market setting (IV) features complete markets, and all five risky basis assets Y_{MKT}, $Y_{HML^{GE}}$, Y_3, Y_4, and Y_5 are admissible.

Within the entangled risk setting (I), we can analyze the different implications of the entangled jump risk $d\mathcal{N}_{1t}$ versus the disentangled jump risk $d\mathcal{N}_{2t}$. The comparison between the entangled risk setting (I) and the disentangled risk settings (II) and (III) further sheds light on the importance of risk entanglement. The fact that both settings feature incomplete markets and jump risks and all of the parameters are identical except for the admissibility of Y_4 demonstrates that market incompleteness and jump risks do not suffice to address the international finance puzzles. At least one jump risk must be entangled to resolve the puzzles.

When we compare settings (I) and (II), we have the same number of admissible assets, and therefore, retain a similar level of market incompleteness, but $Y_{HML^{GE}}$ is missing in setting (II). When we compare settings (I) and (III), we retain $Y_{HML^{GE}}$, but setting (III) has one more admissible asset (i.e., asset Y_4) than setting (I), and thus markets are closer to complete. Accordingly, we analyze both settings (II) and (III) instead of just one of them to demonstrate that the differences between the entangled and disentangled risk settings are not due to the fact that (i) we have more assets in the disentangled risk setting or (ii) $Y_{HML^{GE}}$ is missing from the disentangled setting.[8]

In all four settings Y_3 is an admissible basis asset and jump risk $d\mathcal{N}_{2t}$ is disentangled. Accordingly, the exchange rate risk loading Δ_{e2} on $d\mathcal{N}_{2t}$ is identical across all four settings and is equal to $\Delta_{e2} = \Delta_{H2} - \Delta_{F2} = -0.3637$. A sudden, large devaluation of the home (i.e., the high interest rate) currency implies a carry trade crash. Indeed, we occasionally observe large carry trade losses in the data, and a devaluation of the high interest rate currency by $e^{-0.3637} - 1 = -30.5\%$ appears plausible. Note that it is difficult to exactly measure the jump size and intensity in the data. However, jumps have implications on moments such as the volatility, correlations, (co-)skewness, and kurtosis as well as the entropy. In our discussion below, we compare model-implied moments with estimates in the data to assess whether the risk loadings of the exchange rate are reasonable.

In the two settings with disentangled risks (II) and (III) and the complete market setting (IV), jump risk $d\mathcal{N}_{1t}$ is disentangled as Y_4 is an admissible basis asset. Hence, the exchange rate risk loading Δ_{e1} on $d\mathcal{N}_{1t}$ is identical across these three settings and is equal to $\Delta_{e1} = \Delta_{H1} - \Delta_{F1} = 1.7286$. A devaluation of the foreign (i.e., the low interest rate) currency by $e^{-1.786} - 1 = -82.3\%$ has not occurred in the data that we have analyzed. In contrast, we have $\Delta_{e1} = -0.0595 \neq \Delta_{H1} - \Delta_{F1}$ in setting (I), in which Y_4 is not an admissible basis asset and jump risk $d\mathcal{N}_{1t}$ is entangled with the diffusion risks. A moderate devaluation of the home (i.e., the high interest rate) currency by $e^{-0.0595} - 1 = -5.8\%$ is more reasonable.

Another important observation is that in setting (I), when jump risk $d\mathcal{N}_{1t}$ is entangled with the diffusion risks, the exchange rate loading $\Delta_{e1} = -0.0595$ has the opposite sign of $\Delta_{H1} - \Delta_{F1} = 1.7286$. This finding is interesting because it illustrates that the home currency may depreciate as a result of a shock that increases the home SDF more than the foreign SDF (or even decreases the foreign SDF). This pattern is impossible to generate in a setting with disentangled risks, in which the home currency always appreciates when a jump increases the home SDF more than the foreign SDF. Note, however, that entanglement is a necessary (not a sufficient) condition for the exchange rate to jump in the opposite direction of $\frac{M_{Ht}}{M_{Ft}}$. In Appendix A.2 we present an example where the signs of Δ_{e1} and $\Delta_{H1} - \Delta_{F1}$ are the same, although jump $d\mathcal{N}_{1t}$ is entangled with the diffusion risks.

In the bottom panel of Table 2 we report the model-implied moments based on the risk loadings of Y_{MKT}, $Y_{HML^{GE}}$, and CT_i for all settings $i \in \{I, II, III, IV\}$. All moments are annualized and calculated for returns denominated in the home currency, except for Mean (foreign currency), which is the expected excess return denominated in the foreign currency. A star $*$ indicates that the model-implied moment is outside of the 90% confidence interval of the corresponding estimated moment in Table 1.

[8] In fact, (i) the disentangled setting (II) has the same number of assets as the entangled setting (I), and (ii) $Y_{HML^{GE}}$ is present in both the disentangled setting (III) and the entangled setting (I).

Overall, the model-implied moments of the carry trade return CT_I in the entangled risk setting (I) match the empirical moments well. thirteen out of fourteen model-implied moments of CT_I are within the estimated 90% confidence intervals in Table 1. Coskewness($Y_{HML^{GE}}, Y_{HML^{GE}}, CT_I$) is the only moment that the entangled risk setting struggles to fit well, and yet it still lies within the 96% interval. In comparison, the model-implied carry trade returns do not fit the data in the incomplete market settings with disentangled risks (II) and (III) and in the complete market setting (IV). Most model-implied moments of CT_i for $i \in \{II, III, IV\}$ are outside of the estimated 90% confidence intervals in Table 1. Settings (II) and (III) produce almost identical model-implied moments. Thus, the presence of $Y_{HML^{GE}}$ does not seem to affect our findings when risks are disentangled. In the following we focus on the moments that comprise the international finance puzzles.

First, the model-implied FX premium of CT_I in the home and foreign currencies are 4.8% and 7.3%, which perfectly match the point estimates in the data. In contrast, in settings (II), (III), and (IV), the FX premium in the home currency ranges between -57.2% to -6.4%, and the same premium to the foreign investor ranges between 43.4% to 74.7%. The complete market setting (IV) produces more counterfactual results (i.e., the deviations of the model-implied from the empirical moments are larger) than the incomplete market settings with disentangled risks. To summarize, only setting (I) with entangled jump risk $d\mathcal{N}_{1t}$ generates an FX premium that accurately matches the data.

Second, the volatility of CT_I is 18%, which is slightly higher than the point estimate of 16.7% in the data. However, 18% is well within the 90% confidence interval. In comparison, the model-implied volatility is 38.6% in the incomplete market settings with disentangled risks (II) and (III), and it is 98.3% in the complete market setting (IV). Note that the excess volatility in settings (II), (III), and (IV) is not only due to the large jump risk exposure $\Delta_{e1} = 1.7286$. The diffusion risk loadings of e_I are less than half the magnitude of the loadings of e_{II} and e_{III} and are about an order of magnitude smaller than the loadings of e_{IV}. This translates into a considerably more volatile exchange rate when risks are disentangled than when they are entangled. To summarize, the exchange rate volatility puzzle of Brandt et al. (2006) is prevalent in the disentangled risk settings (II), (III), and (IV), while such a puzzle does not arise in setting (I) in which jump risk $d\mathcal{N}_{1t}$ is entangled.

Third, we analyze the cyclicality of the exchange rate. In Table 1 we demonstrate that the exchange rate is not significantly correlated with the ratio of consumption growths across countries. This is consistent with the findings of Backus and Smith (1993) and Lustig and Verdelhan (2019). As aforementioned, if a country-specific SDF is decreasing in local consumption growth, then our model should generate a low correlation between the exchange rate and the ratio of the SDFs. The correlation between the exchange rate growth $\frac{e_{It+dt}}{e_{It}}$ and the ratio of SDFs $\frac{M_{Ht+dt}/M_{Ht}}{dM_{Ft+dt}/M_{Ft}}$ is zero in setting (I), while it is 86.9% in settings (II) and (III) and is 100% in the complete market setting (IV). The perfect correlation confirms the well-known fact that the exchange rate is equal to the ratio of SDFs in complete markets. Market incompleteness helps to break the rigid relation and reduce the correlation, albeit in our calibration it is lowered only by 13.1% if risks are disentangled. Risk entanglement further reduces the correlation all the way to zero.

Lustig and Verdelhan (2019) argue that the correlation between the exchange rate and the ratio of SDFs may not be the best measure to quantify the exchange rate cyclicality puzzle. Instead, they propose regressing the ratio of the SDFs on the exchange rate growth to obtain the slope coefficient β_{BS} as an additional measure of the exchange rate cyclicality. Similar to the correlation, this regression coefficient is close to zero in the data as Lustig and Verdelhan (2019) report and we confirm in Table 1. However, the regression coefficient sets a higher bar. Lustig and Verdelhan (2019) point out that adding orthogonal risks to the SDFs decreases the correlation between the exchange rate and the ratio of the SDFs, but in a regression such orthogonal risks are irrelevant. They further demonstrate that in a log-normal setting, which is a special case of risk disentanglement (Remark 1), β_{BS} is always equal to one. However, their modeling approach does not produce β_{BS} in closed form for nonlog-normal settings.

A contribution of our analysis is to determine β_{BS} analytically, which is possible because we explicitly model the jump risks that give rise to the nonlog-normal distribution (Appendix B). We compute the regression coefficient β_{BS} for our four settings. We find that $\beta_{BS} = 1$ in disentangled risk settings (II), (III), and (IV). In contrast, in the entangled risk setting (I), we find that β_{BS} is zero, which resolves the exchange rate cyclicality puzzle. This demonstrates the importance of risk entanglement. The jump risk loading Δ_{e1} is a key component to determine the correlation between the exchange rate and the ratio of SDFs and the regression coefficient β_{BS}. The entanglement of the jump risk $d\mathcal{N}_{1t}$ with the diffusion risks flips the sign of Δ_{e1}, which has a negative impact on the correlation between the exchange rate and the ratio of the SDFs.

To summarize, our calibrated model with entangled risks matches 26 out of 28 moments estimated in the data and satisfies the symmetry condition that the home and foreign SDFs have identical volatilities. In particular, setting (I) with entangled risks reconciles the international finance puzzles, namely it generates (i) a smooth exchange rate while SDFs are volatile and low correlated, (ii) a low correlation between the exchange rate and the ratio of SDFs, and (iii) a currency premium that is comparable to average FX returns in the data. In contrast, settings (II) and (III) with disentangled risks in incomplete markets or complete market setting (IV) cannot match most moments and fail to reconcile the international finance puzzles.

3.3. Dissecting the entanglement of jump risk $d\mathcal{N}_{1t}$

To obtain a better understanding of the inner workings of risk entanglement, Table 3 illustrates how the exchange rate in setting (I) changes as we change the set of diffusion risks that are entangled with jump risk $d\mathcal{N}_{t1}$. Table 3 has the same structure as Table 2. We again consider the same jump and diffusion risks, SDFs, and risky basis assets Y_{MKT},

Table 2
Entangled versus disentangled jump risks.

Jump intensities: $\lambda_1 = 0.115$ $\lambda_2 = 0.126$

	SDFs		Basis assets Y_n				
	M_H	M_F	Y_{MKT}	$Y_{HML^{GE}}$	Y_3	Y_4	Y_5
Diffusion dZ_{1t}	0.4519	-0.3786	-0.0768	0.0119	0	0	0.1
Diffusion dZ_{2t}	-0.4110	-0.4810	0.0404	0.0781	0	0	0
Diffusion dZ_{3t}	-0.4459	-0.0330	-0.0840	0.0338	0	0	0
Jump $d\mathcal{N}_{1t}$	0.6501	-1.0785	0.0824	0.0010	0	0.1	0
Jump $d\mathcal{N}_{2t}$	0.5954	0.9590	-0.3500	0.0258	0.1	0	0

	Model-implied exchange rates e for four economies			
	(I)	(II)	(III)	(IV)
	e_I $(MKT, HML^{GE}, 3)$ *Entangled*	e_{II} $(MKT, 3, 4)$ *Disentangled*	e_{III} $(MKT, HML^{GE}, 3, 4)$ *Disentangled*	e_{IV} $(MKT, HML^{GE}, 3, 4, 5)$ *Complete*
Diffusion dZ_{1t}	0.0568	0.1381	0.1385	0.8304
Diffusion dZ_{2t}	-0.0166	-0.0727	-0.0690	0.0701
Diffusion dZ_{3t}	0.0654	0.1511	0.1525	-0.4129
Jump $d\mathcal{N}_{1t}$	-0.0595	1.7286	1.7286	1.7286
Jump $d\mathcal{N}_{2t}$	-0.3637	-0.3637	-0.3637	-0.3637

	Model-implied moments					
	Y_{MKT}	$Y_{HML^{GE}}$	(I) CT_I	(II) CT_{II}	(III) CT_{III}	(IV) CT_{IV}
Mean (home currency)	0.035	0.039	0.048	-0.066*	-0.064*	-0.572*
Mean (foreign currency)			0.073	0.434*	0.436*	0.747*
Volatility	0.162	0.086	0.180	0.386*	0.386*	0.983*
Skewness	-0.738	0.003	-1.824	0.925*	0.925*	0.056*
Kurtosis	4.381	3.001*	7.436	5.574	5.574	3.061*
Entropy	0.015	0.004	0.014	0.137*	0.137*	0.546*
Corr(MKT,y)		-0.111	0.176	-0.029*	-0.029*	-0.012*
Corr(HML^{GE}, y)			0.009	NA	0.002	0.001
Co-skew(MKT,MKT,y)		0.126	-1.023	-0.404	-0.404	-0.158
Co-skew(HML^{GE},HML^{GE}, y)	-0.021		-0.028*	NA	-0.013*	-0.005*
Co-skew(MKT,y, y)			-1.350	-0.020	-0.020	-0.003
Co-skew(HML^{GE}, y, y)			0.226	NA	0.055	0.008
Corr$\left(\frac{M_{Ht}}{M_{Ft}}, y\right)$			0.000	0.869*	0.869*	1*
β_{BS}			0.000	1*	1*	1*
	M_H	M_F				
Volatility	0.867	0.867				
Corr(M_H, y))		0.182				

Notes: Calibration of four economies. There are three diffusion risks ($dZ_{1t}, dZ_{2t}, dZ_{3t}$), and two jump risks ($d\mathcal{N}_{1t}, d\mathcal{N}_{2t}$). Top panel: The panel shows the intensities of jumps $d\mathcal{N}_{1t}, d\mathcal{N}_{2t}$ and risk loadings of the SDFs and basis assets denominated in the home currency. Basis asset Y_{MKT} represents the global stock market and $Y_{HML^{GE}}$ the global high-minus-low book-to-market portfolio. Basis asset Y_3 only loads on jump $d\mathcal{N}_{2t}$ and disentangles jump $d\mathcal{N}_{2t}$. The magnitude of Y_3's loading is irrelevant for our analysis. Similarly, basis assets Y_4 and Y_5 disentangle jump $d\mathcal{N}_{1t}$ and diffusion Z_{1t}. There are 22 parameters (five risk loadings for each $M_H, M_F, Y_{MKT}, Y_{HML^{GE}}$, and two jump intensities). Middle panel: Given the SDFs and the traded basis assets, the exchange rate loadings are derived from equation system (17). Each column provides results of a different economy. For each economy, we only report one pricing-consistent exchange rate that best matches the empirical moments. In the first economy $Y_{MKT}, Y_{HML^{GE}}, Y_3$ are traded, jump $d\mathcal{N}_{1t}$ is entangled with the diffusion risks, and jump $d\mathcal{N}_{2t}$ is disentangled. In the second (third) economy Y_{MKT}, Y_3, Y_4 ($Y_{MKT}, Y_{HML^{GE}}, Y_3, Y_4$) are traded and both jumps are disentangled. In the fourth economy $Y_{MKT}, Y_{HML^{GE}}, Y_3, Y_4, Y_5$ are traded and markets are complete. Bottom panel: Model-implied moments correspond to the empirical moments in Table 1. We assume the home currency has a higher interest rate than the foreign currency, $r_H > r_F$. The carry trade is defined as borrowing e_t units of the foreign and lending one unit of the home currency, $CT = (1 + r_H dt) - (1 + r_F dt)\frac{e_t}{e_{t+dt}}$. Variable y in the correlation and co-skewness terms refers to the corresponding variables in the column heads. A star * indicates that the model-implied moment is outside of the 90% confidence interval of the corresponding estimated moment in Table 1. Formulas for the derivation of the moments are in Appendix B.

$Y_{HML^{GE}}$, Y_3, and Y_5 as in Section 3.2. In addition, we introduce Y_6, which only loads on diffusion dZ_{2t}, and Y_7, which only loads on diffusion dZ_{3t}.

We analyze four different settings. As before, when risks are entangled and there exist multiple pricing-consistent exchange rates, we focus on the solution that best matches the moments in the data. In setting (V) we drop $Y_{HML^{GE}}$, and only Y_{MKT} and Y_3 are admissible risky basis assets. This has two effects on the determination of the pricing-consistent exchange rate when compared to setting

Table 3
Dissecting the entanglement of jump risk $d\mathcal{N}_{1t}$.

Jump intensities: $\lambda_1 = 0.115$ $\lambda_2 = 0.126$

	SDFs				Basis assets Y_n			
	M_H	M_F	Y_{MKT}	$Y_{HML^{GE}}$	Y_3	Y_5	Y_6	Y_7
Diffusion dZ_{1t}	0.4519	-0.3786	-0.0768	0.0119	0	0.1	0	0
Diffusion dZ_{2t}	-0.4110	-0.4810	0.0404	0.0781	0	0	0.1	0
Diffusion dZ_{3t}	-0.4459	-0.0330	-0.0840	0.0338	0	0	0	0.1
Jump $d\mathcal{N}_{1t}$	0.6501	-1.0785	0.0824	0.0010	0	0	0	0
Jump $d\mathcal{N}_{2t}$	0.5954	0.9590	-0.3500	0.0258	0.1	0	0	0

	Model-implied exchange rates e for four economies			
	(V) e_V $(MKT, 3)$ *Entangled*	(VI) e_{VI} $(MKT, HML^{GE}, 3, 4)$ *Entangled*	(VII) e_{VII} $(MKT, HML^{GE}, 3, 5)$ *Entangled*	$(VIII)$ e_{VIII} $(MKT, HML^{GE}, 3, 6)$ *Entangled*
Diffusion dZ_{1t}	0.0554	0.8304	0.4547	0.6288
Diffusion dZ_{2t}	-0.0291	0.1237	0.0701	0.1033
Diffusion dZ_{3t}	0.0606	-0.5328	-0.2746	-0.4129
Jump $d\mathcal{N}_{1t}$	-0.0601	0.6834	-0.7137	-0.4887
Jump $d\mathcal{N}_{2t}$	-0.3637	-0.3637	-0.3637	-0.3637

	Model-implied moments					
	Y_{MKT}	$Y_{HML^{GE}}$	(V) CT_V	(VI) CT_{VI}	(VII) CT_{VII}	$(VIII)$ CT_{VIII}
Mean (home currency)	0.035	0.039	0.041	-0.569*	-0.145*	-0.315*
Mean (foreign currency)			0.066	0.492*	0.220*	0.307*
Volatility	0.162	0.086	0.179	1.020*	0.660*	0.804*
Skewness	-0.738	0.003	-1.839	0.003*	-0.487*	-0.076*
Kurtosis	4.381	3.001*	7.484	3.011*	3.735*	3.054*
Entropy	0.015	0.004	0.013	0.525*	0.191*	0.314*
Corr(MKT,y)		-0.111	0.177	0.043*	-0.028*	0.005*
Corr(HML^{GE}, y)			NA	0.002	0.001	0.001
Co-skew(MKT,MKT,y)		0.126	-1.026	-0.163	-0.327	-0.252
Co-skew(HML^{GE},HML^{GE}, y)	-0.021		NA	-0.005*	-0.008*	-0.006*
Co-skew(MKT,y, y)			-1.357	-0.028	0.050*	-0.031
Co-skew(HML^{GE}, y, y)			NA	0.007	0.020	0.012
Corr$\left(\frac{M_{Ht}}{M_{Ft}}, y\right)$			-0.002	0.754*	0.226*	0.356*
β_{BS}			-0.020	1.306*	0.720*	0.836*
	M_H	M_F				
Volatility	0.867	0.867				
Corr(M_H, y))		0.182				

Notes: Calibration of four economies. There are three diffusion risks ($dZ_{1t}, dZ_{2t}, dZ_{3t}$), and two jump risks ($d\mathcal{N}_{1t}, d\mathcal{N}_{2t}$). Top panel: The panel shows the intensities of jumps $d\mathcal{N}_{1t}, d\mathcal{N}_{2t}$ and risk loadings of the SDFs and basis assets denominated in the home currency. Basis asset Y_{MKT} represents the global stock market and $Y_{HML^{GE}}$ the global high-minus-low book-to-market portfolio. Basis asset Y_3 only loads on jump $d\mathcal{N}_{2t}$ and disentangles jump $d\mathcal{N}_{2t}$. The magnitude of Y_3's loading is irrelevant for our analysis. Similarly, basis assets Y_4, Y_5, and Y_6 disentangle diffusions Z_{1t}, Z_{2t}, and Z_{3t}. There are 22 parameters (five risk loadings for each $M_H, M_F, Y_{MKT}, Y_{HML^{GE}}$, and two jump intensities). Middle panel: Given the SDFs and the traded basis assets, the exchange rate loadings are derived from equation system (17). Each column provides results of a different economy. For each economy, we only report one pricing-consistent exchange rate that best matches the empirical moments. In the first economy Y_{MKT}, Y_3 are traded, jump $d\mathcal{N}_{1t}$ is entangled with the diffusion risks, and jump $d\mathcal{N}_{2t}$ is disentangled. In the second economy $Y_{MKT}, Y_{HML^{GE}}, Y_3, Y_5$ are traded, jump $d\mathcal{N}_{1t}$ is entangled with the diffusion risks Z_{2t} and Z_{3t} but not with Z_{1t}, and jump $d\mathcal{N}_{2t}$ is disentangled. In the third economy $Y_{MKT}, Y_{HML^{GE}}, Y_3, Y_6$ are traded, jump $d\mathcal{N}_{1t}$ is entangled with the diffusion risks Z_{1t} and Z_{3t} but not with Z_{2t}, and jump $d\mathcal{N}_{2t}$ is disentangled. In the fourth economy $Y_{MKT}, Y_{HML^{GE}}, Y_3, Y_7$ are traded, jump $d\mathcal{N}_{1t}$ is entangled with the diffusion risks Z_{1t} and Z_{2t} but not with Z_{3t}, and jump $d\mathcal{N}_{2t}$ is disentangled. Bottom panel: Model-implied moments correspond to the empirical moments in Table 1. We assume the home currency has a higher interest rate than the foreign currency, $r_H > r_F$. The carry trade is defined as borrowing e_t units of the foreign and lending one unit of the home currency, $CT = (1 + r_H dt) - (1 + r_F dt)\frac{e_t}{e_{t+dt}}$. Variable y in the correlation and co-skewness terms refers to the corresponding variables in the column heads. A star * indicates that the model-implied moment is outside of the 90% confidence interval of the corresponding estimated moment in Table 1. Formulas for the derivation of the moments are in Appendix B.

(I). First, more flexibility is gained when choosing the exchange rate as there is one asset less that needs to be correctly priced. That is, if $Y_{HML^{GE}}$ is removed from the economy, one equation can be dropped from system (17). On the other hand, flexibility is lost as $\frac{B_F}{e}$ is now constrained to the span of only the three basis assets B_H, Y_{MKT}, and Y_3 in (16). Notice that independent as to whether asset $Y_{HML^{GE}}$ is included or removed, jump risk $d\mathcal{N}_{1t}$ is entangled and $d\mathcal{N}_{2t}$ is disentangled. Thus, $Y_{HML^{GE}}$ has no effect on the nonlinearity of system (17). We find that the exchange rate loadings in setting (V) are close to the ones in (I). Moreover, the model-implied moments in setting (V) are close to the ones in setting (I) as reported in the bottom panel. We conclude that $Y_{HML^{GE}}$ does not affect the ability to resolve the international finance puzzles. We choose to include $Y_{HML^{GE}}$ in our baseline model (i.e., setting (I)) as it adds ten more moments to our analysis and ensures that there are more moment conditions than model parameters.

In setting (VI), Y_{MKT}, $Y_{HML^{GE}}$, Y_3, and Y_5 are admissible. Therefore, $d\mathcal{N}_{1t}$ is entangled with diffusion risks dZ_{2t} and dZ_{3t} but not with dZ_{1t}. In setting (VII), we replace Y_5 by Y_6, and $d\mathcal{N}_{1t}$ is entangled with diffusion risks dZ_{1t} and dZ_{3t} but not with dZ_{2t}. Finally, in setting $(VIII)$, we replace Y_6 by Y_7, and $d\mathcal{N}_{1t}$ is entangled with diffusion risks dZ_{1t} and dZ_{2t} but not with dZ_{3t}. There are two important observations. First, in all three settings, ten or more (out of fifteen) model-implied moments of CT_i for $i \in \{VI, VII, VIII\}$ are outside of the 90% confidence intervals estimated in Table 1. In particular, the model-implied FX premium is off, the FX volatility is too high, and the correlation between the exchange rate and the ratio of the SDFs is also too high. Thus, it is more difficult to capture the data and reconcile the international finance puzzles when we reduce the degree of entanglement of $d\mathcal{N}_{1t}$ with the diffusion risks.

Second, the deviations of several model-implied moments from their empirical counterparts are largest in setting (VI), second largest in $(VIII)$, and smallest in (VII). In particular, this is true for the FX premium, the FX volatility, $\mathrm{Corr}\left(\frac{M_H}{M_F}, e\right)$, and β_{BS}, i.e., the moments that govern the international finance puzzles. We further observe that the magnitude of the difference in market prices $|\eta_{Fi} - \eta_{Hi}|$ is largest for $i = 1$, second largest for $i = 3$, and smallest for $i = 2$. That is, removing a diffusion dZ_{it} with a relatively large cross-country differential in market prices $|\eta_{Fi} - \eta_{Hi}|$ from the entanglement with the jump risk $d\mathcal{N}_{1t}$ leads to a relatively more significant decrease in the model performance.

We can explain this pattern as follows. According to (17), a large term $\eta_{Fi} - \eta_{Hi} - \sigma_{ei}$ allows for a large deviation of Δ_{e1} from $\Delta_{H1} - \Delta_{F1}$. Of course, the difference between Δ_{H1} and Δ_{F1} has to be sufficiently large; otherwise a deviation of Δ_{e1} from $\Delta_{H1} - \Delta_{F1}$ is not a desirable feature. When Y_5 is admissible and dZ_{1t} is not entangled with $d\mathcal{N}_{1t}$, the entanglement loses a lot of bite because the relatively large term $|\eta_{F1} - \eta_{H1}| = 0.8304$ associated with risk dZ_{1t} vanishes from (17). Accordingly, $\Delta_{e1} = 0.6834$ does not differ significantly from $\Delta_{H1} - \Delta_{F1} = 1.7286$. The introduction of Y_5 has first-order implications and changes the exchange rate jump loading Δ_{e1} from -0.0595 (in setting (I)) to 0.6834 (in setting (VI)), pushing it closer to a setting where $d\mathcal{N}_{1t}$ is disentangled. In comparison the introduction of Y_6 has a weaker effect on (17) as $|\eta_{F2} - \eta_{H2}| = 0.0701$ is relatively small, and $\Delta_{e1} = -0.7137$ deviates significantly from $\Delta_{H1} - \Delta_{F1} = 1.7286$.

To conclude, the key insights are the following. To generate quantitatively relevant results and resolve international finance puzzles, it is not enough that a jump is entangled with other risks. First, it is important that the jump risk features a sufficiently large difference in market prices across the two countries. Second, it is crucial that the jump risk is entangled with other risks that feature a sufficiently large difference in market prices across the two countries. These important insights provide guidance to macro-finance models as to what type of market incompleteness can mitigate international finance puzzles.

4. Conclusion

This paper introduces the concept of risk entanglement in incomplete FX markets to jointly explain the exchange rate volatility, cyclicality, and currency premia. Risk entanglement specifies a subset of incomplete market models in which nondiffusive or nonlog-normal shocks to exchange rates are not fully spanned. Risk entanglement breaks the rigid relation between the exchange rate and the ratio of projected SDFs. In turn, this allows the risk entanglement setting to decouple the exchange rate from consumption dynamics and resolve international finance puzzles. It is important to note that market incompleteness alone is not sufficient to resolve the puzzles. That is, when markets are incomplete and risks are disentangled, the exchange rate is still equal to the ratio of projected SDFs (although it is not necessarily equal to the ratio of the full SDFs), giving rise to several puzzles. Moreover, when risks are entangled, there exist multiple pricing-consistent exchange rates, providing flexibility to obtain an exchange rate that explains empirical facts.

Finally, in a calibration exercise, the paper illustrates the quantitative importance of risk entanglement to match key asset pricing moments in the data. Our calibration exercise provides several interesting insights. To generate quantitatively relevant results and resolve international finance puzzles, it is not enough that a jump is entangled with other risks. It is important that the jump risk features a sufficiently large difference in market prices across the two countries. In addition, it is crucial that this jump risk is entangled with other risks that also feature a sufficiently large difference in market prices across the two countries. These are important insights that provide guidance to macro-finance models as to what type of market incompleteness can mitigate international finance puzzles.

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Appendix A. Sensitivity analysis

This appendix analyzes how the exchange rate solution varies with different inputs of the model considered in the main text.

A1. Sensitivity analysis: market price of $d\mathcal{N}_{1t}$

Fig. B.1 depicts the sensitivity of our results in the entangled risk setting (I) to changes in either the market price Δ_{H1} (black solid line) or Δ_{F1} (red dashed line) while keeping constant all other parameters. The top left graph illustrates how the correlation between the home and foreign SDFs depends upon the market prices. The correlation is decreasing in the difference $\Delta_{H1} - \Delta_{F1}$. That is, either an increase in Δ_{F1} or a decrease in Δ_{H1}, while holding the other market prices constant, increases the correlation.

The top right graph illustrates how the jump size of the exchange rate Δ_{e1} is affected. Keeping all other parameters constant, according to (17) an increase in $\Delta_{H1} - \Delta_{F1}$ likely increases Δ_{e1}. A caveat to this intuition is that σ_{ed} for $d \in \{1, 2, 3\}$ may adjust as we change $\Delta_{H1} - \Delta_{F1}$. As such, the final effect on Δ_{e1} is ambiguous. However, if the changes in σ_{ed} are moderate for $d \in \{1, 2, 3\}$, then Δ_{e1} is increasing in $\Delta_{H1} - \Delta_{F1}$. Our calibration confirms this intuition. Starting with the parameters in setting (I), an increase in Δ_{F1} or a decrease in Δ_{H1} increases the magnitude of Δ_{e1}. Moreover, if we increase Δ_{H1}, then Δ_{e1} increases and approaches zero.

The graphs in the middle show the sensitivity of the FX volatility (graph on the left) and the entropy (graph on the right). These quantities are increasing in the magnitude of the jump size Δ_{e1}. Accordingly, they are decreasing in $\Delta_{H1} - \Delta_{F1}$. The bottom left graph displays the sensitivity of the carry trade premium from the perspective of home investors. While Δ_{F1} has an effect on the jump size Δ_{e1}, the jump risk $d\mathcal{N}_{1t}$ does not have a high market price from the perspective of home investors. Thus, the response of the carry trade premium to changes in Δ_{F1} is small. In contrast, changes in Δ_{H1} have a large effect. This is because Δ_{H1} affects both the jump size Δ_{e1} and the market price of $d\mathcal{N}_{1t}$ from the perspective of home investors. The premium is increasing in Δ_{H1}. The marginal increase is becoming smaller as an increase in Δ_{H1} also decreases the magnitude in Δ_{e1}.

Finally, the bottom right graph illustrates how β_{BS} depends on Δ_{H1} and Δ_{F1}. The intuition for this relation is more difficult. Changes in Δ_{H1} or Δ_{F1}, from the parametrization in setting (I), affect the exchange rate's loadings on the diffusion risks through (17) and increase the correlation between the exchange rate and the ratio of the SDFs. This increase in the correlation increases β_{BS}. Taking the other model parameters as given, the graph suggests that a large difference between Δ_{H1} and Δ_{F1} is necessary to generate a low β_{BS}.

To summarize, setting (I) relies on a sufficiently large difference in the market prices Δ_{H1} and Δ_{F1} associated with the entangled jump risk $d\mathcal{N}_{1t}$.

A2. Sensitivity analysis: calibration with only positive jumps in the SDFs

In the previous analysis, the foreign SDF loads negatively on jump risk $d\mathcal{N}_{1t}$, suggesting that a jump indicates a good state from the foreign perspective. In contrast, the home SDF has a positive exposure and jump risk $d\mathcal{N}_{1t}$ is a bad state to home investors. While such a scenario is possible, we now search our parameter space for different values that satisfy $\Delta_{Hi} > 0$ and $\Delta_{Fi} > 0 \forall i \in \{1, 2\}$.

Table A.4 presents a calibration where home and foreign SDFs have positive exposures to all jump risks, that is, the jumps are considered bad states from both home and foreign perspectives. Table A.4 is structured the same as Table 2, and we compare again four settings. In setting (IX), jump risk $d\mathcal{N}_{1t}$ is entangled, while $d\mathcal{N}_{2t}$ is disentangled. In settings (X) and (XI), both jump risks are disentangled but in (X) $Y_{HML^{GE}}$ is not admissible, while it is in setting (XI). Markets are complete in setting (XII).

As in the baseline calibration in Table 2, we are able to resolve the international finance puzzles only when risks are entangled in Table A.4. Specifically, setting (IX) reconciles the international finance puzzles and generates (i) a smooth exchange rate while SDFs are volatile and low correlated, (ii) a low correlation between the exchange rate and the ratio of SDFs, and (iii) a currency premium that is comparable to average FX returns in the data. Note that setting (IX) does not fit the data as well as setting (I) in our baseline calibration in Table 2. It particularly struggles to match the empirical co-skewness. Nevertheless, in setting (IX), 22 model-implied moments are within the estimated 90% confidence interval and the model satisfies the symmetry condition that home and foreign SDFs have the same volatility.

In comparison, settings (X), (XI), and (XII) with disentangled risks have several more moments that are inconsistent with the data. In particular, they fail to reconcile the international finance puzzles. The FX premium is off, the FX volatility is twice (or more) than what we estimate in the data, the exchange rate correlates too much with the ratio of SDFs, and $\beta_{BS} = 1$.

Finally, a notable difference between the risk entanglement settings (I) and (IX) is that in setting (IX), the sign of Δ_{e1} is the same as the sign of $\Delta_{H1} - \Delta_{F1}$, while these signs are opposite in setting (I).

Appendix B. Technical details and proofs

This appendix provides details for the portfolio representation of the exchange rate in the general jump diffusion setting of Section 2.2 and expressions for the key asset pricing quantities underlying the calibration of Section 3.

B1. Portfolio representation of the exchange rate

Let e_t denote the amount of foreign currency that buys one unit of the home currency at t. The tradability of the foreign bond (16) and asset return specifications (14) imply

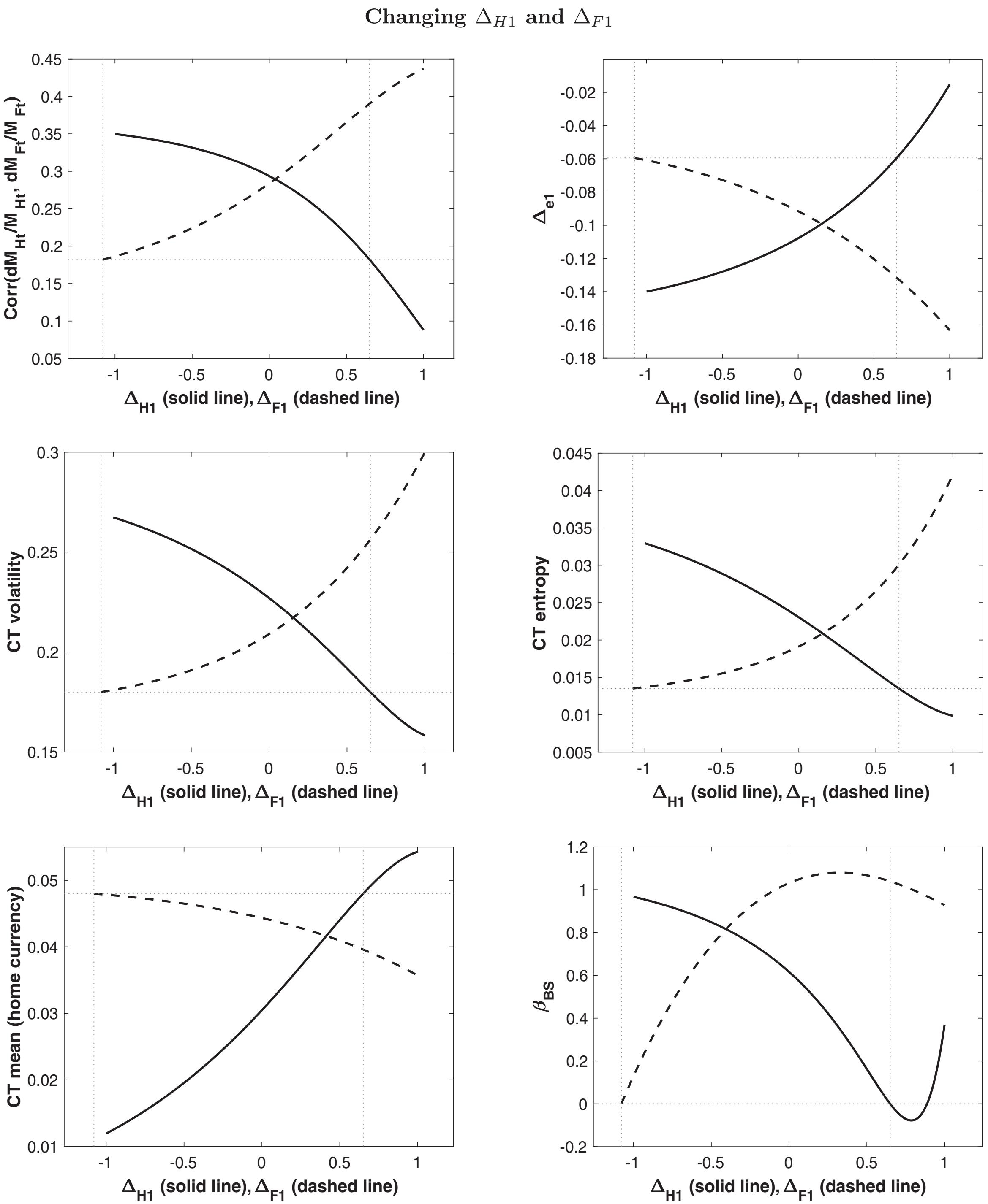

Fig. B.1. The plots illustrate how changes in either the loadings of the home (Δ_{H1}; black line) or foreign (Δ_{F1}; dashed line) SDF on the entangled jump risk $d\mathcal{N}_{1t}$ affect various quantities of interest in setting (I) of Table 2 while keeping all other parameters constant. The quantities of interest are the correlation between the home and foreign SDFs ($Corr(dM_{Ht}/M_{Ht}, dM_{Ft}/M_{Ft})$; top left panel), the loading of the exchange rate on the entangled jump risk $d\mathcal{N}_{1t}$ (Δ_{e1}; top right panel), the volatility of the currency carry trade (CT volatility; middle left panel), the entropy of the currency carry trade (CT entropy; middle right panel), the average currency carry trade denominated in the home currency (CT mean (home currency); bottom left panel), and the Backus-Smith regression coefficient (β_{BS}, bottom right panel). The vertical and horizontal dotted lines indicate the parameter values of Δ_{H1} and Δ_{F1} in setting (I).

Table A1
Imposing positive jumps in the SDFs.

Jump intensities: $\lambda_1 = 0.134$ $\lambda_2 = 0.059$

	SDFs				Basis assets Y_n		
	M_H	M_F	Y_{MKT}	$Y_{HML^{GE}}$	Y_3	Y_4	Y_5
Diffusion dZ_{1t}	0.4744	-0.2346	0.0487	0.0028	0	0	0.1
Diffusion dZ_{2t}	0.4965	0.2626	-0.0040	-0.0537	0	0	0
Diffusion dZ_{3t}	-0.4882	-0.1828	0.1259	0.0142	0	0	0
Jump $d\mathcal{N}_{1t}$	0.4584	1.1156	0.0082	-0.1599	0	0.1	0
Jump $d\mathcal{N}_{2t}$	0.1134	0.5560	-0.3735	0.2242	0.1	0	0

	Model-implied exchange rates e for four economies			
	(IX) e_{IX} $(MKT, HML^{GE}, 3)$ *Entangled*	(X) e_X $(MKT, 3, 4)$ *Disentangled*	(XI) e_{XI} $(MKT, HML^{GE}, 3, 4)$ *Disentangled*	(XII) e_{XII} $(MKT, HML^{GE}, 3, 4, 5)$ *Complete*
Diffusion dZ_{1t}	-0.0199	-0.0130	0.0016	0.7090
Diffusion dZ_{2t}	-0.0760	0.0011	0.2695	0.2339
Diffusion dZ_{3t}	-0.0415	-0.0336	-0.0307	-0.3054
Jump $d\mathcal{N}_{1t}$	-0.1978	-0.6573	-0.6573	-0.6573
Jump $d\mathcal{N}_{2t}$	-0.4426	-0.4426	-0.4426	-0.4426

	Model-implied moments					
	Y_{MKT}	$Y_{HML^{GE}}$	(IX) CT_{IX}	(X) CT_X	(XI) CT_{XI}	(XII) CT_{XII}
Mean (home currency)	0.042	0.042	0.048	0.066	-0.073*	-0.525*
Mean (foreign currency)			0.073	0.139*	0.073	0.198*
Volatility	0.155	0.099*	0.181	0.369*	0.456*	0.886*
Skewness	-0.483	0.527	-1.968	-2.360	-1.245	-0.170*
Kurtosis	3.972	6.149	8.613	8.745	5.448	3.172*
Entropy	0.013	0.005*	0.014	0.044*	0.080*	0.369*
Corr(MKT,y)		-0.175	0.149	0.077	0.062	0.032*
Corr(HML^{GE}, y)			-0.030	NA	-0.105	-0.054
Co-skew(MKT,MKT,y)		0.610*	-0.740	-0.364	-0.294	-0.151
Co-skew(HML^{GE},HML^{GE}, y)	-0.756*		-1.544*	NA	-1.080*	-0.556*
Co-skew(MKT,y, y)			-1.121	-0.227	-0.148	-0.039
Co-skew(HML^{GE}, y, y)			1.136*	NA	-0.609*	-0.161*
Corr$\left(\frac{M_{Ht}}{M_{Ft}}, y\right)$			-0.000	0.241*	0.404*	1*
β_{BS}			-0.000	1*	1*	1*
	M_H	M_F				
Volatility	0.870	0.870				
Corr(M_H, y))		0.362				

Notes: Calibration of four economies. There are three diffusion risks ($dZ_{1t}, dZ_{2t}, dZ_{3t}$), and two jump risks ($d\mathcal{N}_{1t}, d\mathcal{N}_{2t}$). Top panel: The panel shows the intensities of jumps $d\mathcal{N}_{1t}, d\mathcal{N}_{2t}$ and risk loadings of the SDFs and basis assets denominated in the home currency. Basis asset Y_{MKT} represents the global stock market and $Y_{HML^{GE}}$ the global high-minus-low book-to-market portfolio. Basis asset Y_3 only loads on jump $d\mathcal{N}_{2t}$ and disentangles jump $d\mathcal{N}_{2t}$. The magnitude of Y_3's loading is irrelevant for our analysis. Similarly, basis assets Y_4 and Y_5 disentangle jump $d\mathcal{N}_{1t}$ and diffusion Z_{1t}. There are 22 parameters (five risk loadings for each $M_H, M_F, Y_{MKT}, Y_{HML^{GE}}$, and two jump intensities). Middle panel: Given the SDFs and the traded basis assets, the exchange rate loadings are derived from equation system (17). Each column provides results of a different economy. For each economy, we only report one pricing-consistent exchange rate that best matches the empirical moments. In the first economy $Y_{MKT}, Y_{HML^{GE}}, Y_3$ are traded, jump $d\mathcal{N}_{1t}$ is entangled with the diffusion risks, and jump $d\mathcal{N}_{2t}$ is disentangled. In the second (third) economy Y_{MKT}, Y_3, Y_4 ($Y_{MKT}, Y_{HML^{GE}}, Y_3, Y_4$) are traded and both jumps are disentangled. In the fourth economy $Y_{MKT}, Y_{HML^{GE}}, Y_3, Y_4, Y_5$ are traded and markets are complete. Bottom panel: Model-implied moments correspond to the empirical moments in Table 1. We assume the home currency has a higher interest rate than the foreign currency, $r_H > r_F$. The carry trade is defined as borrowing e_t units of the foreign and lending one unit of the home currency, $CT = (1 + r_H dt) - (1 + r_F dt)\frac{e_t}{e_{t+dt}}$. Variable y in the correlation and co-skewness terms refers to the corresponding variables in the column heads. A star * indicates that the model-implied moment is outside of the 90% confidence interval of the corresponding estimated moment in Table 1. Formulas for the derivation of the moments are in Appendix B.

an exchange rate process,[9] with the exchange rate's drift, diffusion, and jump components:

$$\mu_e = r_F - \left[1 - \sum_{Y\in\{Y\}} \alpha_Y\right] r_H + \sigma_e'\sigma_e - \sum_{Y\in\{Y\}} \alpha_Y \left[\mu_Y - \sum_{i\in\mathcal{J}_Y} \lambda_i\left(e^{\Delta_{Yi}} - 1\right)\right], \tag{B.1}$$

$$\sigma_e = -\sum_{Y\in\{Y\}} \alpha_Y\sigma_Y, \quad e^{\Delta_{ei}} \equiv \frac{1}{1 + \sum_{Y\in\mathcal{Y}_i} \alpha_Y\left(e^{\Delta_{Yi}} - 1\right)}, \quad \forall i \in \mathcal{J}_{\{Y\}}. \tag{B.2}$$

Above, $\mathcal{J}_{\{Y\}}$ denotes the set of all jump types impacting asset returns $\{Y\}$ and $\mathcal{Y}_i$ the set of all assets impacted by the jump type i.

B2. Asset pricing quantities

We present expressions for several asset pricing quantities in the general jump diffusion setting. The main text employs various special (simplified) versions of these general expressions. First, the market integration assumption (Assumption 1) implies that a traded asset Y can be priced in any currency denomination as follows

$$E_t\left[\frac{M_{Ht+dt}}{M_{Ht}}\frac{Y_{t+dt}}{Y_t}\right] = E_t\left[\frac{M_{Ft+dt}}{M_{Ft}}\frac{e_{t+dt}}{e_t}\frac{Y_{t+dt}}{Y_t}\right] = 1, \quad \forall Y \in \{Y\}. \tag{B.3}$$

Moments of asset returns: We consider a generic jump diffusion growth process

$$\frac{X_{t+dt}}{X_t} = 1 + \mu_X dt + \sigma_X' dZ_t + \sum_i \left(e^{\Delta_{Xi}} - 1\right)(d\mathcal{N}_{it} - \lambda_i dt) \tag{B.4}$$

as well as an asset return process $\frac{Y_{t+dt}}{Y_t}$ (14) denominated in the home currency in particular.

The annualized conditional mean and volatility of generic process X are

$$\frac{1}{dt}E_t\left[\frac{X_{t+dt}}{X_t} - 1\right] = \mu_X, \quad \sqrt{\frac{1}{dt}Var_t\left[\frac{X_{t+dt}}{X_t} - 1\right]} = \sqrt{|\sigma_X|^2 + \sum_i \lambda_i\left(e^{\Delta_{Xi}} - 1\right)^2}. \tag{B.5}$$

In case of asset returns, the expected excess return in the home currency can be computed using Euler pricing Eq. (B.3),

$$Mean\ (home\ currency)(Y) = \eta_H'\sigma_Y - \sum_{i=1} \lambda_i\left(e^{\Delta_{Hi}} - 1\right) \times \left(e^{\Delta_{Yi}} - 1\right). \tag{B.6}$$

The annualized return volatility is

$$Volatility(Y) = \sqrt{|\sigma_Y|^2 + \sum_i \lambda_i\left(e^{\Delta_{Yi}} - 1\right)^2}. \tag{B.7}$$

With the volatility in (B.5), the skewness and kurtosis of generic process X are, respectively, (Ait-Sahalia, 2004)

$$Skewness(X) = \frac{\sum_{i=1} \lambda_i\left(e^{\Delta_{Xi}} - 1\right)^3}{Volatility(X)^3}, \tag{B.8}$$

$$Kurtosis(X) = \frac{\sum_{i=1}^2 \lambda_i\left(e^{\Delta_{Xi}} - 1\right)^4}{Volatility(X)^4} + 3. \tag{B.9}$$

The annualized covariance and correlation between X and Y are, respectively,

$$Cov(X,Y) = \sigma_X'\sigma_Y + \sum_i \lambda_i\left(e^{\Delta_{Xi}} - 1\right)\left(e^{\Delta_{Yi}} - 1\right), \tag{B.10}$$

$$Corr(X,Y) = \frac{Cov(X,Y)}{Volatility(X)Volatility(Y)}. \tag{B.11}$$

The co-skewness of X, X, and Y is

$$Co\text{-}Skew(X,X,Y) = \frac{\sum_i \lambda_i\left(e^{\Delta_{Xi}} - 1\right)^2\left(e^{\Delta_{Yi}} - 1\right)}{Volatility(X)^2 Volatility(Y)}. \tag{B.12}$$

Volatilities and correlation of the exchange rate and SDF ratio: The total conditional variances (including both jump and diffusion contribution) of the SDF (13) and the exchange rate (15) are, respectively,

$$\frac{1}{dt}Var_t\left(\frac{M_{It+dt}}{M_{It}}\right) = |\eta_{It}|^2 + \sum_{i\in\mathcal{J}_I}\left(e^{\Delta_{Iit}} - 1\right)^2\lambda_{it}, \quad \forall I, \tag{B.13}$$

$$\frac{1}{dt}Var_t\left(\frac{e_{t+dt}}{e_t}\right) = |\sigma_e|^2 + \sum_{i\in\mathcal{J}_{\{Y\}}} \lambda_{it}\left(e^{\Delta_{ei}} - 1\right)^2, \tag{B.14}$$

where σ_e, Δ_{ei} are functions of asset return moments (B.2), and therefore exchange rate jump types $\mathcal{J}_{\{Y\}}$ are from asset returns. The total conditional covariances between SDFs of home and foreign countries and their exchange rate are, respectively,

$$\frac{1}{dt}Cov_t\left(\frac{M_{Ht+dt}}{M_{Ht}}, \frac{M_{Ft+dt}}{M_{Ft}}\right) = \eta_{Ht}'\eta_{Ft} + \sum_{i\in(\mathcal{J}_H\cap\mathcal{J}_F)} \lambda_{it}\left(e^{\Delta_{Hit}} - 1\right)\left(e^{\Delta_{Fit}} - 1\right), \tag{B.15}$$

$$\frac{1}{dt}Cov_t\left(\frac{\frac{M_{Ht+dt}}{M_{Ht}}}{\frac{M_{Ft+dt}}{M_{Ft}}}, \frac{e_{t+dt}}{e_t}\right) = (-\eta_{Ht} + \eta_{Ft})'\sigma_e + \sum_{i\in(\mathcal{J}_{\{Y\}}\cap\mathcal{J}_H\cap\mathcal{J}_F)} \lambda_{it}\left(e^{\Delta_{Hit} - \Delta_{Fit}} - 1\right)\left(e^{\Delta_{ei}} - 1\right). \tag{B.16}$$

Currency premia: Consider a zero-net currency strategy of borrowing e_t units of the foreign and lending one unit the home currency from t to $t + dt$. The realized return denominated in the home currency is

$$CT^H_{t+dt} = (1 + r_H dt) - (1 + r_F dt)\frac{e_t}{e_{t+dt}} = (r_H - r_F)dt - (1 + r_F dt)\left(\frac{e_t}{e_{t+dt}} - 1\right). \tag{B.17}$$

[9] Substituting (14) into (16) only yields an expression for the reciprocal of the exchange rate growth $\frac{e_t}{e_{t+dt}}$. We apply Itô's lemma to yield the multiplicative inverse of this ratio to obtain the proper exchange rate growth $\frac{e_{t+dt}}{e_t}$.

The risk premium from the home perspective is

$$\text{Mean (home)}(CT) = -\frac{1}{dt}Cov_t\left(\frac{dM_{Ht+dt}}{M_{Ht}}, CT^H_{t+dt}\right) = \eta'_H\sigma_e + \sum_{i\in(\mathcal{J}_H\cap\mathcal{J}_{\{Y\}})}\lambda_i\left(e^{\Delta_{Hi}}-1\right)\left(e^{-\Delta_{ei}}-1\right). \quad (B.18)$$

Similarly, consider a zero-net currency strategy of borrowing one unit of the foreign and lending $\frac{1}{e_t}$ units of the home currency. The realized return on this currency strategy is

$$CT^F_{t+dt} = (1+r_H dt)\frac{e_{t+dt}}{e_t} - (1+r_F dt) = (r_H - r_F)dt + (1+r_H dt)\left(\frac{e_{t+dt}}{e_t}-1\right), \quad (B.19)$$

and the risk premium in the foreign currency denomination is

$$\text{Mean (foreign)}(CT) = -\frac{1}{dt}Cov_t\left(\frac{dM_{Ft+dt}}{M_{Ft}}, CT^F_{t+dt}\right) = \eta'_F\sigma_e - \sum_{i\in(\mathcal{J}_F\cap\mathcal{J}_{\{Y\}})}\lambda_i\left(e^{\Delta_{Fi}}-1\right)\left(e^{\Delta_{ei}}-1\right). \quad (B.20)$$

Backus-Smith measure of cyclicality: As defined in Lustig and Verdelhan (2019), this measure is obtained (as the slope coefficient) by linearly regressing the ratio of a pair of pricing kernels on the exchange rate. In our jump diffusion setting, the Backus-Smith measure of cyclicality is

$$\beta_{BS} \equiv \frac{Cov_t\left(\frac{\frac{M_{Ht+dt}}{M_{Ht}}}{\frac{M_{Ft+dt}}{M_{Ft}}}, \frac{e_{t+dt}}{e_t}\right)}{Var_t\left(\frac{e_{t+dt}}{e_t}\right)} = \frac{(-\eta_{Ht}+\eta_{Ft})'\sigma_e + \sum_{i\in(\mathcal{J}_{\{Y\}}\cap\mathcal{J}_H\cap\mathcal{J}_F)}\lambda_{it}\left(e^{\Delta_{Hit}-\Delta_{Fit}}-1\right)\left(e^{\Delta_{ei}}-1\right)}{|\sigma_e|^2 + \sum_{i\in\mathcal{J}_{\{Y\}}}\lambda_{it}\left(e^{\Delta_{ei}}-1\right)^2}. \quad (B.21)$$

Entropy formulas: Consider the generic growth process (B.4). The annualized entropy of this growth process is defined and computed as follows

$$Entropy(X) \equiv L_t\left(\frac{X_{t+dt}}{X_t}\right) \equiv \frac{1}{dt}\left\{\log\left(E_t\left[\frac{X_{t+dt}}{X_t}\right]\right) - E_t\left[\log\left(\frac{X_{t+dt}}{X_t}\right)\right]\right\}$$

$$= \frac{1}{2}|\sigma_X|^2 + \sum_i \lambda_i\left(\left[e^{\Delta_{Xi}}-1\right]-\Delta_{Xi}\right). \quad (B.22)$$

In case the jump sizes Δ_{Xi} are sufficiently small such that the exponential function can be replaced by its convergent expansion series, the entropy is equal to the limit of the following convergent series

$$L_t\left(\frac{X_{t+dt}}{X_t}\right) = \frac{1}{2}|\sigma_X|^2 + \sum_i \lambda_i \sum_{k=2}\frac{(\Delta_{Xi})^k}{k!}. \quad (B.23)$$

In addition, the conditional variance of the growth process

$$\frac{1}{dt}Var_t\left[\log\left(\frac{X_{t+dt}}{X_t}\right)\right] = |\sigma_X|^2 + \sum_i \lambda_i\left(e^{\Delta_{Xi}}-1\right)^2, \quad (B.24)$$

is equal to the entropy (B.22) multiplied by two (and up to order $(\Delta_{Xi})^2$). However, when jump sizes are large, higher orders of Δ_{Xi} are important, and the conditional variance deviates from the entropy (B.22).

Proof of Theorem 1: The key equation system determining the exchange rate is (17), which is established by combining Euler pricing equations of the risk-free bonds and the risky assets in both currency denominations. In the first direction, we assume that risks in asset markets are completely disentangled and derive the uniqueness of the exchange rate solution. To isolate diffusion risks, we apply Eq. (17) on each of assets $\{X_k\}$, $k\in\{1,\ldots,d\}$, that load only on a diffusion risk (as risks are disentangled in markets, such assets can always be constructed from traded assets)

$$\eta_{Hk} - \eta_{Fk} + \sigma_{ek} = 0. \quad (B.25)$$

As a result, the diffusion of exchange rate is uniquely determined from the prices of diffusion risks in the home and foreign countries.

To isolate jump risks, we consider in turn a jump type i that affects

a. both home and foreign SDFs ($i\in\mathcal{J}_Y\cap\mathcal{J}_H\cap\mathcal{J}_F$). Applying Eq. (17) on each asset W_i that loads only on this jump type i yields $\lambda_i e^{\Delta_{Fi}+\Delta_{ei}}\left(e^{\Delta_{Wi}}-1\right) = \lambda_i\left(e^{\Delta_{Hi}}-1\right)\left(e^{\Delta_{Wi}}-1\right) + \lambda_i\left(e^{\Delta_{Wi}}-1\right)$. Canceling common factor $\lambda_i\left(e^{\Delta_{Wi}}-1\right)$ from both sides determines uniquely the exchange rate's jump size with respect to type i, $e^{\Delta_{ei}}-1 = e^{\Delta_{Hi}-\Delta_{Fi}}-1$, $\forall i\in\mathcal{J}_Y\cap\mathcal{J}_H\cap\mathcal{J}_F$;
b. the home but not the foreign SDF ($i\in\mathcal{J}_Y\cap\mathcal{J}_H\setminus\mathcal{J}_F$). Applying Eq. (17) on each asset W_i that loads only on this jump type i determines uniquely the exchange rate's respective jump size, $e^{\Delta_{ei}}-1 = e^{\Delta_{Hi}}-1$, $\forall i\in\left(\mathcal{J}_{\{Y\}}\cap\mathcal{J}_H\right)\setminus\mathcal{J}_F$;
c. the foreign but not the home SDF ($i\in\mathcal{J}_Y\cap\mathcal{J}_F\setminus\mathcal{J}_H$). Applying Eq. (17) on each asset W_i that loads only on this jump type i determines uniquely the exchange rate's respective jump size, $e^{\Delta_{ei}}-1 = e^{-\Delta_{Fi}}-1$, $\forall i\in\left(\mathcal{J}_{\{Y\}}\cap\mathcal{J}_F\right)\setminus\mathcal{J}_H$;
d. neither the home nor the foreign SDF ($i\in\mathcal{J}_Y\setminus\mathcal{J}_H\setminus\mathcal{J}_F$). Applying Eq. (17) on each asset W_i that loads only on this jump type i determines uniquely the exchange rate's respective jump size, $\Delta_{ei}=0$, $\forall i\in\left(\mathcal{J}_{\{Y\}}\setminus\mathcal{J}_H\right)\setminus\mathcal{J}_F$.

Technically, this uniqueness of the exchange rate arises from a property that in the presence of risk disentanglement, (17) is completely decoupled into separate equations, and each equation involves only one type of risk. As a result, (17) can also be cast into an equation system that is linear in exchange rate weights $\{\alpha_Y\}$ and determines a unique exchange rate solution. In the opposite direction, in the presence of risk entanglement, (17) remains a system of coupled equations that are nonlinear in exchange rate weights, giving rise to multiple pricing-consistent exchange rate solutions in general ■

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