Recovery and Consistency

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Abstract

Recovery, the process of unambiguously determining market's beliefs, time and risk preferences from asset prices, requires a subjective state space specification of the underlying market that is not observed prior to the recovery implementation. Two different subjective specifications lead to two distinct sets of recovery results. While these results are unique under their respective specifications, they are generally inconsistent with each other. These inconsistencies vary significantly across specifications and vanish only when the underlying marginal utilities and probabilities are identical for consolidating states that map one specification to the other. Using option price data, we empirically demonstrate that recovery inconsistencies are prevalent, significant, and robust. Our findings indicate that a consistent recovery framework remains elusive.

Keywords: Recovery, Inconsistency, Specification.

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1 Introduction

The prospect of inferring rational characteristics of market participants, such as their risk preferences, time preferences and expectations (i.e., beliefs), from the prices of financial assets has remained elusive due to a fundamental ambiguity. Specifically, different combinations of preferences and beliefs can be compatible with the same set of asset prices. As a result, prices alone are insufficient to unambiguously recover a unique set of market participants' characteristics. Ross (2015)'s Recovery Theorem addresses this challenge by identifying specific conditions under which a unique recovery is possible. While conceptually elegant, the theorem's practical applicability depends critically on whether these unique characteristics can be reliably obtained under the identified conditions.

To shed light on this question, our paper introduces and integrates the consistency requirement into the recovery framework. The importance of recovery consistency is motivated by a fundamental observation: a specification of the underlying market's state space (e.g., the number of states) is a necessary input for the recovery process. However, not only such a state space specification is unobserved prior to the recovery implementation, but also it is highly consequential for the recovery results. The recovery consistency requirement posits that different sets of recovered characteristics, each uniquely obtained under a subjective specification, be reconcilable, as they all pertain to the same (true) market participants.

We derive a necessary and sufficient condition for consistent recoveries while maintaining the premise of the Recovery Theorem (i.e., upholding its assumptions). Specifically, given two different subjective specifications employed in the recovery process, this condition requires that the underlying characteristics (marginal utilities and transition probabilities) be identical for the states that differentiate the two specifications. Intuitively, when these differentiating states are associated with indistinguishable underlying characteristics, switching between the two specifications does not result in any information loss about the underlying market, thereby preserving recovery consistency. Empirically, we document robust recovery inconsistencies and demonstrate the model-implied variation between recovery inconsistencies and state space specification inputs using option data.

Evidently, the above necessary and sufficient condition for recovery consistency is highly restrictive, implying that different subjective specifications most likely yield inconsistent recovery results. The key insight and practical value of this consistency condition lie precisely in its failure, namely, that its violation informs our findings on (i) the source and (ii) the scope of recovery inconsistencies. To investigate these recovery inconsistency aspects in a general and tractable approach, our paper analyzes a perturbative setup in which the subjective specifications of two analysts deviate slightly from the consistency condition. Specifically, we consider two subjective specifications: a fine one characterized by more states and a coarse one by fewer states. We then examine the differences in the associated recovery results and analyze their behavior when the coarse specification becomes finer.[1] This perturbative setup is instrumental in delineating two key groups of components: the unperturbed (consistent) components, which preserve information about the underlying characteristics, and the perturbative (inconsistent) components, which distort this information and contribute to recovery inconsistencies.

The perturbative analysis offers two important insights into the nature of recovery inconsistencies. First, since the unperturbed (leading-order) components are consistent, the next-order terms, which couple the unperturbed and perturbative components, are responsible for recovery inconsistencies. These couplings, expressed as scalar products (i.e., uncentered covariances) of eigenvectors pertaining to the unperturbed and perturbative components, demonstrate that the source of recovery inconsistencies is grounded in the co-variation between these components. Intuitively, these couplings allow the inconsistencies and information losses intrinsic to the perturbative components to spill over and distort the overall

[1] In our empirical analysis using option data, the domain of moneyness (i.e., the ratio of strike price to spot price) represents the state space. The recovery implementation involves imposing a grid structure on this state space, where the number of nodes on the grid (and the grid's step size) determines the state space specification. The choice of the grid structure, and thus the (subjective) state space specification, may be influenced or constrained by the computing power available to individual analysts in practice.

recovery results obtained by the two analysts.

Second, recovery inconsistencies generally persist even when the coarse specification approaches, but remains distinct from, the fine one.[2] Intuitively, because the perturbative components are unconstrained by the consistency requirement, they are exogenous to (i.e., independent of) the unperturbed consistent components. Given this exogeneity, refining the subjective specification does not necessarily alter their alignment (co-variation or couplings) in a definitive direction, thereby preventing a universal improvement in recovery consistency.

Given the premise of the Recovery Theorem, its key analytical result is that the recovered time and risk preferences are exclusively characterized by the dominant eigenvalue and right eigenvector of the Arrow-Debreu state price transition matrix (or AD price matrix). Our analysis extends this result by demonstrating that recovery inconsistencies are characterized by eigenvalues and both right and left eigenvectors of all orders, as well as their couplings. This insight helps to deliver analytical expressions for recovery inconsistencies across generic specifications of any dimensions in a perturbative analysis. Although this setup is a simplified framework for tractability purposes, it is indicative and insightful about a profound consistency issue in real-world recovery settings. As long as a generic (i.e., non-perturbative) recovery setting can be partitioned into consistent and inconsistent components, their co-variation remains a key determinant of recovery inconsistencies.

Our empirical investigation is guided by the conceptual insights of the recovery consistency condition and perturbative analysis to evaluate recovery inconsistencies. We do not assume or impose a consistent or perturbatively consistent structure on the data. Our empirical analysis consists of three parts.

First, we collect prices for spot and call and put options on the S&P 500 (from January 5, 2000, to August 30, 2023, with maturities ranging from 20 to 730 days) and follow literature's neural network approach to estimate the entire implied volatility (IV) surface for each date

[2] By construction, recovery inconsistencies vanish only when the perturbative (inconsistent) components vanish, which is not the limit of our interest. Instead, holding the non-vanishing perturbative components fixed, we examine whether the refinement of the specification input improves recovery consistency.

in our sample. The resulting IV surface is smooth and stable, allowing for the construction of AD price matrices of different dimensions, which are essential for our recovery consistency analysis.

Second, we implement separate recovery processes for 601-state (fine) and 150-state (coarse) specifications. A comparative analysis of the recovery results across these specifications reveals the prevalence and significance of inconsistencies in the recovered transition probabilities and risk preferences.

Third, we formulate several model-implied measures to quantify violations of the recovery consistency condition in the data, such as the divergence of marginal utilities and transition probabilities across fine states that map into a single coarse state. We estimate positive and statistically significant co-variations between the level of recovery inconsistencies and the magnitude of these recovery consistency violation measures. These estimates provide empirical support for a key conceptual thesis: recovery inconsistencies increase when specification inputs are less consistent. Our empirical results are robust to different sample periods, measures of recovery consistency, and specifications.

Related literature: There has been a long-standing interest in the recovery of market's beliefs, risk and time preferences from traded asset prices. Breeden and Litzenberger (1978) derive the risk-neutral distribution conditional on a (fictitious) null risk aversion using option prices. Ross (2015)'s Recovery Theorem identifies sufficient conditions, i.e., assumptions of time-separable preferences and homogeneous transition probabilities, to disentangle market's beliefs from risk and time preferences unambiguously. The literature evaluates the empirical performance of the Recovery Theorem by examining the validity of its assumptions. Borovička et al. (2016) and Hansen and Scheinkman (2017) relate these assumptions to the dominance and recoverability of the transitory component of Alvarez and Jermann (2005)'s stochastic discount factor (SDF) growth decomposition. Bakshi et al. (2018), Qin et al. (2018), and Jackwerth and Menner (2020) provide evidence of counterfactual asset pricing implications arising from these assumptions. Our paper complements this literature by demonstrating that consistent recoveries remain elusive even when the Recovery Theorem's assumptions are taken as given.

Our empirical implementation of the recovery process closely follows the approach of Audrino et al. (2021), who employ a neural network approach as in Ludwig (2015) and regularization techniques to obtain stable implied volatility surfaces and AD price matrices from sparse option price data. Building on this literature, we extend the empirical analysis to investigate recovery consistency by implementing and comparing separate recoveries under different specifications. We relate observable price data inputs across these recoveries using the law of one price, hence preserving no-arbitrage conditions and analysts' subjective choices in selecting specifications. This approach allows us to demonstrate recovery inconsistencies across different specifications.

Another strand of literature aims to generalize the Recovery Theorem by extending its premise or relaxing its restrictive assumptions. Carr and Yu (2012) derive the recovery in a continuous setting with bounded underlying stochastic state variables. Walden (2017) generalizes the recovery to settings with an unbounded support for state dynamics. Qin and Linetsky (2016) extend Ross's recovery to continuous-time Markov processes. Dillschneider and Maurer (2019) discuss the Perron-Frobenius operator theory in recovery. Martin and Ross (2019) examine the relationship between recovered time preferences and the unconditional expected return on long-maturity bonds. Jensen et al. (2019) propose a generalized recovery approach by relaxing the time-homogeneity assumption and accommodating a growing state space. In a related context of principal eigenproblem in asset pricing, Borovička and Stachurski (2020) derive a necessary and sufficient condition for the existence of a unique and stable value function for a class of recursive utilities. Employing a tractable perturbative setup, our analysis complements these studies by reaching beyond the Perron-Frobenius dominant eigenvalue and positive eigenvector of the AD price matrix systematically. The analysis highlights the importance of eigenvalues and both left and right eigenvectors of all orders in quantifying recovery inconsistencies.

Our findings suggest that surveys remain the direct and informative channel for learning about market's beliefs and their rich and complex contingent investment decisions as in Giglio et al. (2022). In a model-free (non-parametric) approach, asset price data can only provide some bounds on market's expectations, as shown by Martin (2017) and Gormsen and Koijen (2020).

The current paper is organized as follows. Section 2 introduces the consistency requirement and derives the consistency condition for the recovery framework. Section 3 identifies the recovery consistency issue with loss of price information in recovery implementations in a general setting and quantifies this relationship in a tractable perturbative analysis, also drawing insights from a parallel continuous setting. Section 4 demonstrates the prevalence and significance of recovery inconsistencies in the data. Section 5 concludes. Online Appendices A and B present further empirical evidence, methodological details and technical derivations.

2 Recovery and Consistency Requirement

This section introduces the setting, motivations, and notations for the paper's subsequent analysis of the consistency issue in recovery. We begin by briefly presenting the underlying assumptions and implementation of Ross (2015)'s Recovery Theorem. We formulate the concept of recovery consistency in Section 2.1 and obtain a complete (necessary and sufficient) condition to characterize the recovery consistency issue in Section 2.2.

2.1 Basic Recovery Framework

Recovery setup and assumptions: The basic recovery framework in a discrete setting starts with a standard specification of the underlying (data-generating) finite state space and time,

$$\text{State index: } i \in \mathcal{S} \equiv \{1, \ldots, S\}, \qquad \text{time index: } t \in \{0, \ldots, T\}. \tag{1}$$

We assume that the associated financial market is complete and free of arbitrage opportunities. As a result, a unique stochastic discount factor (SDF) process exists and can be identified with the marginal utility of the representative agent in the economy. To uniquely determine the representative agent's risk and time preferences, as well as the state probability distribution under the physical measure, from asset prices, Ross (2015)'s recovery framework makes two important assumptions.

Assumption A1. *The preference is time-separable, i.e., the SDF growth $M_{t,t+1}(i,j)$ from the time state (t,i) to $(t+1,j)$ has the following functional form: $M_{t,t+1}(i,j) = \delta \frac{M_j}{M_i}$, $\forall i,j \in \mathcal{S}$, $\forall t \in \{0,\ldots,T-1\}$, where δ is a constant parameter and M_k depends only on the state k, $\forall k \in \mathcal{S}$.*

Assumption A2. *The state transition dynamics are time-homogeneous, i.e., the transition probabilities from the time state (t,i) to $(t+1,j)$ under the physical measure are time-independent: $p_{t,t+1}(i,j) = p_{ij}$, $\forall i,j \in \mathcal{S}$, $\forall t \in \{0,\ldots,T-1\}$.*

The first assumption associates the representative agent's discount factor (i.e., time preference) with a constant δ, and marginal utility (i.e., risk preference) with the state-contingent M_i, for $i \in \mathcal{S}$, in the recovery process. The second assumption associates the underlying state transition with a Markovian recurrent dynamics, allowing for asset prices observed over various time horizons (i.e., tenors) to implicate this transition dynamics.

Recovery Theorem: The two assumptions above give rise to a decomposition of the state-contingent Arrow-Debreu (AD) price matrix in terms of the matrices of transition probabilities and marginal utilities,

$$\mathbf{A} = \mathbf{Diag}\left(\frac{1}{M}\right) \mathbf{P}\, \mathbf{Diag}\,(M)\,, \tag{2}$$

where the $S \times S$ matrix $\mathbf{P}$ contains one-period transition probabilities $\{p_{ij}\}$ under the physical measure, and the diagonal matrix $\mathbf{Diag}\,(M)$ (resp., $\mathbf{Diag}\left(\frac{1}{M}\right)$) contains marginal utilities $\{M_i\}$ (resp., $\left\{\frac{1}{M_i}\right\}$) on the diagonal, $i,j \in \mathcal{S}$. The $S \times S$ matrix $\mathbf{A}$ contains one-period AD asset prices $\{A_{ij}\}$, with $A_{ij} = \delta p_{ij}\frac{M_j}{M_i}$, $i, j \in \mathcal{S}$, denoting the price at the time state (t, i) of the AD asset that offers a unit payoff at $(t+1, j)$ (and zero payoff otherwise). Throughout, boldface is used to denote a vector or a matrix. It is important to note that the above decomposition holds separately for every individual eigenspace of matrices $\mathbf{A}$ and $\mathbf{P}$,

$$\delta^{(k)} = \delta\,\delta_p^{(k)}, \quad \mathbf{x}^{(k,R)} = \mathbf{Diag}\left(\frac{1}{M}\right)\mathbf{p}^{(k,R)}, \quad \mathbf{x}^{(k,L)} = \mathbf{p}^{(k,L)}\mathbf{Diag}\,(M)\,, \quad k \in \mathcal{S}, \tag{3}$$

where scalar $\delta^{(k)}$, $S \times 1$ vector $\mathbf{x}^{(k,R)}$ and $1 \times S$ vector $\mathbf{x}^{(k,L)}$ denotes respectively the k-th eigenvalue, k-th right and left eigenvectors of the one-period AD price matrix, $\mathbf{A}\,\mathbf{x}^{(k,R)} = \delta^{(k)}\,\mathbf{x}^{(k,R)}$, and $\mathbf{x}^{(k,L)}\,\mathbf{A} = \delta^{(k)}\,\mathbf{x}^{(k,L)}$, $k \in \{1, \ldots, S\}$. Similar notations apply for the one-period transition probability matrix, $\mathbf{P}\,\mathbf{p}^{(k,R)} = \delta_p^{(k)}\,\mathbf{p}^{(k,R)}$, and $\mathbf{p}^{(k,L)}\,\mathbf{P} = \delta_p^{(k)}\,\mathbf{p}^{(k,L)}$, $k \in \{1, \ldots, S\}$, and δ is the time discount factor (Assumption A1). These decompositions at the individual eigenspace level succinctly capture key results of the recovery framework.

The dominant right eigenvectors of $\mathbf{P}$ and $\mathbf{A}$ deliver the Recovery Theorem. Since every stochastic matrix $\mathbf{P}$ has a unique constant right eigenvector $\mathbf{p}^{(1,R)} = (1, \ldots, 1)'$ associated with the largest eigenvalue $\delta_p^{(1)} = 1$, the AD price matrix $\mathbf{A}$ also has a unique positive right eigenvector associated with the dominant eigenvalue $\delta^{(1)} = \delta$, which then implies the entire transition probability matrix under the physical measure from (3)

$$\mathbf{A}\mathbf{x}^{(1,R)} = \delta\mathbf{x}^{(1,R)}, \qquad \text{with } x_i^{(1,R)} = \frac{1}{M_i}, \quad \forall i \in \{1, \ldots, S\}. \tag{4}$$

$$\mathbf{P} = \delta^{-1}\mathbf{Diag}\,(M)\,\mathbf{A}\mathbf{Diag}\left(\frac{1}{M}\right), \qquad \text{or} \qquad p_{ij} = \delta^{-1}A_{ij}\frac{M_i}{M_j}, \quad \forall i \in \{1, \ldots, S\}. \tag{5}$$

These unique and positive quantities $\{\delta, M_i, p_{ij}\}$ associated with AD price matrix's dominant eigenspace constitute Ross (2015)'s Recovery Theorem.

The dominant left eigenvectors concern the long-term yield of the market model. Since the dominant left eigenvector $\mathbf{p}^{(1,L)}$ of the stochastic matrix $\mathbf{P}$ represents the steady-state (long-term, unconditional) probability distribution under the physical measure, the AD price matrix's dominant left eigenvector $\mathbf{x}^{(1,L)}$ (3) represents the steady-state probability distribution under the risk-neutral measure (being the product of the physical probability distribution and marginal utility). As a result, $\mathbf{x}^{(1,L)}$ plays an important role in the asymptotic pricing of long-term bonds and yields (Martin and Ross (2019)).

Furthermore, all sub-dominant eigenvalues and right and left eigenvectors of the AD price matrix are crucial for the current paper's recovery consistency analysis. This is because when the underlying state space is unobserved and misspecified, the spectrum of the AD price matrix quantifies the divergence of recovery results from the underlying model's characteristics (Sections 3 below).

Recovery implementation: While the AD price matrix $\mathbf{A}$ is central to the recovery framework and its implementation, we do not fully observe it.[3] Instead, given the time-homogeneous state transition dynamics (Assumption A2), $\mathbf{A}$ is implied from the prices of assets associated with different tenors. Let $A_{\tau;ij}$ be the current price at (t,i) of the τ-period AD asset that offers a unit payoff in the future time state $(t+\tau,j)$ (and zero payoff otherwise). Note that the pricing of $\tau+1$-period AD assets is recursive by rolling the τ-period AD prices an additional period, $A_{\tau+1;ij} = \sum_{k=1}^{S} A_{\tau;ik}A_{kj}$, or in matrix notation[4]

$$\underbrace{\mathbf{A}_{\tau+1}}_{S\times S} = \underbrace{\mathbf{A}_{\tau}}_{S\times S}\underbrace{\mathbf{A}}_{S\times S} \qquad \Longrightarrow \qquad \mathbf{A} = \mathbf{A}_{\tau}^{-1}\,\mathbf{A}_{\tau+1}. \tag{6}$$

Matrix $\mathbf{A}$, obtained from the observable price matrices $\mathbf{A}_\tau$ and $\mathbf{A}_{\tau+1}$ through this procedure, is then employed in Equation (4) to recover the underlying market model.[5]

[3] At the current time state (t,i), we do not observe the price A_{kj} of AD assets initiated in states k different from the current state i of the financial market.

[4] The S rows of $S\times S$ matrix $\mathbf{A}_\tau$ (resp., matrix $\mathbf{A}_{\tau+1}$) contain the current AD prices of S tenors in $\{1,\ldots,S\}$ (resp., S tenors in $\{2,\ldots,S+1\}$).

[5] Given S states, collecting AD prices for $S+1$ different tenors (i.e., $\tau \in \{1,\ldots,S+1\}$ collected for $\mathbf{A}_\tau$ and $\mathbf{A}_{\tau+1}$) produces a linear system (6) to determine $\mathbf{A}$. When the number of available tenors is larger than needed (i.e., more equations than unknowns), $\mathbf{A}$ can be solved from the system (6) via a least-squares approach. In principle, since one row of matrix $\mathbf{A}$ (the one containing the prices of AD assets initiated in the current state) is observed, we only need to collect AD prices for S different tenors (Appendix B.2.1). In practice, our empirical analysis employs options of all tenors in our sample (least squares, Section 4). The collection of S, $S+1$, or more tenors does not alter the nature of the recovery consistency issue, as it is determined by the specification employed in the recovery process (as we elaborate in subsequent sections).

2.2 Consistency Requirement

The recovery process takes as input the parameters of the underlying specification, such as the number of states S or the state distribution in the state space. However, these parameters are not observed prior to the recovery process. As a result, the state space specification becomes a subjective input to the recovery implementation, giving rise to an important question about potential impacts of the specification choice on recovery results. We approach and quantify these impacts by positing on the consistency requirement, which helps to assesses how well different sets of recovery results, recovered respectively and uniquely under different subjective state space specifications, are consistent with each other.

Recovery consistency setup: To illustrate and study the role of the state space specification in the consistency of recovery results, consider a thought experiment setup with two different analysts recovering the same (objective and unobserved) underlying market model. We assume that the first analyst adopts $\mathcal{S} = \{1, \ldots, S\}$ and the second adopts $\overline{\mathcal{S}} = \{\overline{1}, \ldots, \overline{S}\}$ as their respective subjective specification input for the recovery process.

Without loss of generality, let $\overline{S} < S$. We refer to $\mathcal{S}$ and $\overline{\mathcal{S}}$ as the original and consolidated specification, respectively, and employ an overbar to denote quantities associated with the consolidated specification.[6] To simplify the exposition, let the state partition of the consolidated specification $\overline{\mathcal{S}} = \{\overline{1}, \ldots, \overline{j}, \ldots, \overline{S}\}$ be nested within the original specification $\mathcal{S} = \{1, \ldots, j, k, \ldots, S\}$, i.e., $\overline{\mathcal{S}} \subset \mathcal{S}$. Specifically, each state $\overline{j} \in \overline{\mathcal{S}}$ either (i) coincides with a single original state or (ii) contains (couples) several original states from $\mathcal{S}$,

$$\forall \overline{j} \in \overline{\mathcal{S}} : \begin{cases} \text{Single state :} & \overline{j} = j \in \mathcal{S}, \\ \text{Coupled state :} & \overline{j} = \{j, k, \ldots\} \subset \mathcal{S}. \end{cases} \tag{7}$$

[6]The recovery consistency issue, and hence the thought experiment, just requires the two specifications to be different to examine the consistency of their recovery results. The assumption that the first analyst's (original) specification coincides with the underlying $\mathcal{S}$ is made for convenience, as it helps to quantify the recovery inconsistency from the underlying characteristics (preferences and probabilities) in a thought-experiment setting. Importantly, the recovery consistency issue does not require that the underlying specification and characteristics be observed. Our empirical analysis of recovery consistency (Section 4) does not assume knowledge of the underlying (true) specification.

This nesting structure of state partitions, $\overline{\mathcal{S}} \subset \mathcal{S}$, aims to model the feature that one specification (the finer $\mathcal{S}$) accommodates relatively more details about the state space structure than the other specification (the coarser $\overline{\mathcal{S}}$). Within this nested framework, the current states $i \in \mathcal{S}$ (perceived by the first analyst) and $\bar{i} \in \overline{\mathcal{S}}$ (by the second analyst) satisfy $i \in \bar{i}$.

The first analyst determines the one-period AD price matrix $\mathbf{A}$ from Equation (6) and recovers the original market model $\{\delta, \mathbf{M}, \mathbf{P}\}$ characterized by the unique dominant eigenspace of $\mathbf{A}$ (4), where $\mathbf{M} = \{M_i\}$ denotes the set of marginal utilities. In contrast, the second analyst employs the set of current prices $\{\overline{A}_{\tau;\bar{i}\,\bar{j}}\}$ of consolidated τ-period AD assets, in which $(t, \bar{i})$ and $(t+\tau, \bar{j})$ denote respectively the current and payoff time state perceived under the specification $\overline{\mathcal{S}}$. These prices are related to the original AD prices by the law of one price, $\overline{A}_{\tau;\bar{i}\,\bar{j}} = \sum_{j \in \bar{j}} A_{\tau;i\,j}$, $\forall \bar{j} \in \overline{\mathcal{S}}$, and hence are observable. After stacking the consolidated AD prices into $\overline{S} \times \overline{S}$ matrices $\overline{\mathbf{A}}_\tau$ and $\overline{\mathbf{A}}_{\tau+1}$, the second analyst solves for the one-period consolidated AD price matrix $\overline{\mathbf{A}}$ from[7]

$$\underbrace{\overline{\mathbf{A}}_{\tau+1}}_{\overline{S}\times\overline{S}} = \underbrace{\overline{\mathbf{A}}_{\tau}}_{\overline{S}\times\overline{S}} \underbrace{\overline{\mathbf{A}}}_{\overline{S}\times\overline{S}} \quad \Longrightarrow \quad \overline{\mathbf{A}} = \overline{\mathbf{A}}_{\tau}^{-1}\, \overline{\mathbf{A}}_{\tau+1}, \tag{8}$$

and recovers the model $\{\bar{\delta}, \overline{\mathbf{M}}, \overline{\mathbf{P}}\}$ characterized by the unique dominant eigenspace of matrix $\overline{\mathbf{A}}$, where $\overline{\mathbf{M}} = \{\overline{M}_{\bar{i}}\}$.

Consistent recoveries: Given that a single underlying model drives asset prices in the market, recovery results obtained by different analysts are subject to the consistency requirement that these results pertain to the same underlying market model. When this requirement is violated, the two sets of recovered results are inconsistent with each other, indicating that at least one of them is also inconsistent with the underlying market model. For an illustration, Example 1 (Equation (9)) below presents the consistency requirement in a simple recovery

[7]For the recovery process based on the specification $\overline{\mathcal{S}}$, matrix $\overline{\mathbf{A}}_\tau$ stacks the observable consolidated current AD prices $\{\overline{A}_{\tau;\bar{i}\,\bar{j}}\}$ for target states $\bar{j} \in \overline{\mathcal{S}}$ and tenors $\tau \in \{1, \ldots, \overline{S}\}$, and $\overline{\mathbf{A}}_{\tau+1}(\varepsilon)$ for tenors $\tau \in \{2, \ldots, \overline{S}+1\}$. They are similar to $\mathbf{A}_\tau$ and $\mathbf{A}_{\tau+1}$ in (6), which are constructed based on the original specification $\mathcal{S}$.

setting. Before stating a general result, we quantify the notion of consistency among state space specifications for a given underlying market model of time and risk preferences and transition probabilities.

Definition 1 (Consistent specifications) *Let $\mathcal{S} \supset \overline{\mathcal{S}}$ be two nesting state space specifications (7). $\mathcal{S}$ and $\overline{\mathcal{S}}$ are consistent specifications with respect to the recovery if all single states $j \in \mathcal{S}$ belonging to a coupled state $\overline{j} \in \overline{\mathcal{S}}$ are associated with identical marginal utilities and transition probabilities in the underlying model: $M_i = M_k$ and $p_{i\overline{h}} = p_{k\overline{h}}$, $\forall i, k \in \overline{j}$ and $\overline{j}, \overline{h} \in \overline{\mathcal{S}}$.*

The following general result establishes the crucial role of consistent specifications in a consistent recovery (with a complete proof presented in Appendix B.1).

Proposition 1 (Consistent recoveries) *The Ross's recovery results obtained under two different state space specifications are mutually consistent if and only if these specifications satisfy Definition 1.*

This proposition concerns several subtle aspects of the recovery framework. Conceptually, it highlights an endogeneity issue inherent to the recovery process, in which the state space specification is not only needed and unobserved before the recovery implementation but also highly consequential to the recovery results. Furthermore, different subjective specifications can lead to irreconcilable recovery results, even though they are unique to respective analysts (Recovery Theorem). This observation motivates the consistency as an important criterion to qualify a recovery process.

Intuitively, Proposition 1's necessary and sufficient condition requires that underlying states must be indistinguishable before they can be consolidated into a single effective state that produces consistent recovery results. This condition is grounded in an informational aspect. Indistinguishable underlying states imply no loss of information about the underlying market model when consolidating a finer specification $\mathcal{S}$ into a coarser $\overline{\mathcal{S}}$. Consequently, recoveries based on $\mathcal{S}$ and $\overline{\mathcal{S}}$ are consistent.

Quantitatively, this necessary and sufficient condition is highly restrictive and can only be satisfied for some special underlying market models and subjective specifications. Proposition 1's restrictive condition indicates an elusive nature of a consistent recovery. That is, in most settings, recovery results obtained by different analysts are inconsistent with one another. Section 4.3 below documents robust recovery inconsistencies and presents empirical evidence for the relationship between specification inputs and recovery inconsistencies that underlies Proposition 1.

Before presenting qualitative analysis and empirical evidence for the robust consistency issue in the recovery framework, it is instructive to illustrate Proposition 1 with a simple example.

Example 1 (Recovery consistency illustration) *Consider two consistent (original and consolidated as per Definition 1) specifications* $\mathcal{S} = \{1, 2, 3\}$ *and* $\overline{\mathcal{S}} = \{\overline{1}, \overline{2}\}$ *with a nesting relationship* $\overline{\mathcal{S}} \subset \mathcal{S}$ *(7), where* $\overline{1} = 1$ *is the current state and* $\overline{2} = \{2, 3\}$.

Analysts' recovery processes: The thought experiment setup starts with the market model $\{\delta, M_i, p_{ij}\},\ i, j \in \{1, 2, 3\}$, which determines every asset price observable in the model. Endowed with specification $\mathcal{S}$, the first analyst obtains $\mathbf{A} = \mathbf{A}_\tau^{-1}\mathbf{A}_{\tau+1}$ (6), solves its unique dominant eigenspace, and by construction, recovers the underlying model $\{\delta, M_i, p_{ij}\}$. The second analyst observes and employs the τ-period consolidated AD asset prices associated with $\overline{\mathcal{S}} = \{\overline{1}, \overline{2}\}$ and then solves for the 2×2 one-period AD price $\overline{\mathbf{A}}$ (8) and its dominant eigenspace to recover $\{\overline{\delta}, \overline{M}_{\overline{i}}, \overline{p}_{\overline{i}\,\overline{j}}\}$, $\overline{i}, \overline{j} \in \{\overline{1}, \overline{2}\}$.[8]

Recovery consistency: The recovery consistency concerns the compatibility of characteristics $\{\delta, M_i, p_{ij}\}$ and $\{\overline{\delta}, \overline{M}_{\overline{i}}, \overline{p}_{\overline{i}\,\overline{j}}\}$ recovered by the two analysts. As the recovered characteristics are from the same underlying market model, their consistency requirement is intuitive,

$$\overline{\delta} = \delta, \qquad \overline{M}_{\overline{1}} = M_1, \qquad \overline{p}_{\overline{1}\,\overline{1}} = p_{11}, \qquad \overline{p}_{\overline{1}\,\overline{2}} = p_{12} + p_{13}. \tag{9}$$

[8] The law of one price relates the τ-period AD asset prices employed by the two analysts, $\overline{A}_{\tau;\overline{1}\,\overline{1}} = A_{\tau;11}$ and $\overline{A}_{\tau;\overline{1}\,\overline{2}} = A_{\tau;12} + A_{\tau;13}$.

Three key and generalizable aspects of the recovery consistency that Example 1 aims to illustrate are in order.

First, the consistency of specifications $\mathcal{S}$ and $\overline{\mathcal{S}}$ translates into the consistency of the associated one-period AD price matrices $\mathbf{A}$ and $\overline{\mathbf{A}}$. Specifically, given Definition 1, we have a consistent consolidation of $\mathbf{A}$ into $\overline{\mathbf{A}}$ as follows[9]

$$\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \longrightarrow \underbrace{\begin{bmatrix} A_{11} & A_{12}+A_{13} \\ A_{21} & A_{22}+A_{23} \\ A_{31} & A_{32}+A_{33} \end{bmatrix}}_{\equiv \mathbf{A}^{+}} = \begin{bmatrix} \overline{A}_{\overline{1}\,\overline{1}} & \overline{A}_{\overline{1}\,\overline{2}} \\ \overline{A}_{\overline{2}\,\overline{1}} & \overline{A}_{\overline{2}\,\overline{2}} \\ \overline{A}_{\overline{2}\,\overline{1}} & \overline{A}_{\overline{2}\,\overline{2}} \end{bmatrix} \longrightarrow \begin{bmatrix} \overline{A}_{\overline{1}\,\overline{1}} & \overline{A}_{\overline{1}\,\overline{2}} \\ \overline{A}_{\overline{2}\,\overline{1}} & \overline{A}_{\overline{2}\,\overline{2}} \end{bmatrix} = \overline{\mathbf{A}}, \tag{10}$$

where the auxiliary AD matrix $\mathbf{A}^{+}$ is constructed by summing the 2nd and 3rd columns of $\mathbf{A}$, following the consolidation of $\mathcal{S}$ into $\overline{\mathcal{S}}$. When these specifications are consistent, states 2 and 3 are indistinguishable (Definition 1). AD assets initiated in these states have same prices, resulting in identical, and hence redundant, 2nd and 3rd rows of $\mathbf{A}^{+}$. This redundancy signifies the consistency of the consolidation of $\mathcal{S}$ and $\overline{\mathcal{S}}$. Specifically, without any loss of information, the redundant 3rd row of $\mathbf{A}^{+}$ can be dropped, reducing it to the 2×2 AD price matrix $\overline{\mathbf{A}}$ employed in the second analyst's recovery process.[10]

Second, consistent AD price matrices give rise to consistent dominant eigenvalues and eigenvectors. The redundancy of the 2nd and 3rd rows of the auxiliary matrix $\mathbf{A}^{+}$ (10) implies identical 2nd and 3rd components, $x_2^{(1,R)} = x_3^{(1,R)}$, of the original dominant right eigenvector $\mathbf{x}^{(1,R)}$. These identical components effectively transform the matrix product $\mathbf{A}\mathbf{x}^{(1,R)}$ equivalently into $\overline{\mathbf{A}}\,\overline{\mathbf{x}}^{(1,R)}$, in which the 2nd and 3rd columns of $\mathbf{A}$ are consolidated

[9]Substituting the AD asset pricing equations (5) for $\mathcal{S}$ and $\overline{\mathcal{S}}$ into Definition 1 implies $\overline{A}_{\overline{1}\,\overline{1}} = A_{11}$, $\overline{A}_{\overline{2}\,\overline{1}} = A_{21} = A_{31}$, $\overline{A}_{\overline{1}\,\overline{2}} = A_{12} + A_{13}$, and $\overline{A}_{\overline{2}\,\overline{2}} = A_{22} + A_{23} = A_{32} + A_{33}$. Stacking these equations yields the matrix equation (10).

[10] Throughout, $\mathbf{X}^{+}$ denotes the matrix constructed by consolidating (summing) relevant columns $j \in \overline{j}$ (those pertaining to a coupled state $\overline{j} \in \overline{\mathcal{S}}$), separately for every $\overline{j} \in \overline{\mathcal{S}}$, of a generic matrix $\mathbf{X}$. Similarly, $\mathbf{X}_{-}$ is constructed by dropping redundant (identical or extra) rows $j \in \overline{j}$ (those pertaining to a coupled state $\overline{j} \in \overline{\mathcal{S}}$), separately for each $\overline{j} \in \overline{\mathcal{S}}$, of matrix $\mathbf{X}$.

(summed) into the 2nd column of $\overline{\mathbf{A}}$, in accordance with $\{2,3\} = \overline{2}$,

$$\underbrace{\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x_1^{(1,R)} \\ x_2^{(1,R)} \\ x_2^{(1,R)} \end{bmatrix}}_{\mathbf{x}^{(1,R)}} = \delta \underbrace{\begin{bmatrix} x_1^{(1,R)} \\ x_2^{(1,R)} \\ x_2^{(1,R)} \end{bmatrix}}_{\mathbf{x}^{(1,R)}} \longrightarrow \underbrace{\begin{bmatrix} A_{11} & A_{12}+A_{13} \\ A_{21} & A_{22}+A_{23} \end{bmatrix}}_{\overline{\mathbf{A}}} \underbrace{\begin{bmatrix} x_1^{(1,R)} \\ x_2^{(1,R)} \end{bmatrix}}_{\overline{\mathbf{x}}^{(1,R)}} = \delta \underbrace{\begin{bmatrix} x_1^{(1,R)} \\ x_2^{(1,R)} \end{bmatrix}}_{\overline{\mathbf{x}}^{(1,R)}}. \tag{11}$$

Evidently, the dominant right eigenvector $\overline{\mathbf{x}}^{(1,R)} = \left[\overline{x}_1^{(1,R)}, \overline{x}_2^{(1,R)}\right]'$ of the consolidated AD price matrix $\overline{\mathbf{A}}$ coincides with the corresponding part of the original $\mathbf{x}^{(1,R)}$, i.e., $\overline{x}_1^{(1,R)} = x_1^{(1,R)}$ and $\overline{x}_2^{(1,R)} = x_2^{(1,R)}$. Since the dominant right eigenvector of the AD price matrix represents the inverse of recovered marginal utilities, Equation (11) demonstrates the consistency of recovery results obtained by two different analysts, i.e., $\overline{M}_{\overline{1}} = M_1$ and $\overline{M}_{\overline{2}} = M_2 = M_3$.

Third, the dominant left eigenvectors of AD price matrices $\mathbf{A}$ and $\overline{\mathbf{A}}$ are related by $\overline{x}_1^{(1,L)} = x_1^{(1,L)}$ and $\overline{x}_2^{(1,L)} = x_2^{(1,L)} + x_3^{(1,L)}$.[11] Intuitively, these relationships reflect the consistency of the steady-state risk-neutral distributions (as discussed below (5)). Technically, these relationships are a direct consequence of the consistency between AD price matrices.[12] These consistency relationships between the left eigenvectors of AD price matrices are important in our subsequent analysis of recovery inconsistencies when the underlying specification is not observed.

In summary, Example 1 explicitly illustrates the thesis of Proposition 1: when two state space specifications satisfy Definition 1, the associated one-period AD price matrices are consistent, leading to consistent dominant eigenspaces (i.e., eigenvalues and right and left eigenvectors) and consequently consistent recovery results. Next, we formalize all aspects of this consistency sequence, which helps characterize its breakdown and identify the sources

[11] Recall the notation of the dominant left eigenvectors are $\mathbf{x}^{(1,L)} = [x_1^{(1,L)} \; x_2^{(1,L)} \; x_3^{(1,L)}]$ and $\overline{\mathbf{x}}^{(1,L)} = [\overline{x}_1^{(1,L)} \; \overline{x}_2^{(1,L)}]$.

[12] Specifically, summing 2nd and 3rd columns of the original left eigenequation, $\mathbf{x}^{(1,L)}\,\mathbf{A} = \delta\mathbf{x}^{(1,L)}$, yields $\mathbf{x}^{(1,L)}\,\mathbf{A}^+ = \delta[\mathbf{x}^{(1,L)}]^+$, using the notation of Footnote 10. Given the consistency condition that the 2nd and 3rd rows of the auxiliary matrix $\mathbf{A}^+$ are identical (10) and dropping the redundant 3rd row, this equation reduces to $\overline{\mathbf{x}}^{(1,L)}\,\overline{\mathbf{A}} = \delta\overline{\mathbf{x}}^{(1,L)}$, with $\overline{\mathbf{x}}^{(1,L)} = [x_1^{(1,L)} \quad x_2^{(1,L)} + x_3^{(1,L)}]$. This establishes that $\overline{\mathbf{x}}^{(1,L)}$ is the dominant left eigenvector of $\overline{\mathbf{A}}$ as well as its relationships to $\mathbf{x}^{(1,L)}$.

of recovery inconsistencies in a general and robust setting where Proposition 1's restrictive condition does not hold.

3 Recovery and Inconsistency

Given the restrictive premise of a consistent recovery, this section analyzes the alternative prevalent premise where Proposition 1's condition fails and inconsistent recovery arises. Section 3.1 introduces several characterizations of recovery inconsistencies. Section 3.2 demonstrates the source and generality of inconsistent recoveries via a perturbative analysis and connections to the recovery in a continuous setting. Appendix B.3 supplements these qualitative findings with a quantitative analysis.

3.1 General Characterizations

The basic insight gained from the recovery illustration of Example 1 is that different analysts recover consistent results only when their respective one-period AD price matrices are mutually consistent. Let $\mathbf{A}_{S\times S}$ and $\overline{\mathbf{A}}_{\overline{S}\times\overline{S}}$ be the one-period AD price matrices associated with nesting state space specifications $\mathcal{S} \supset \overline{\mathcal{S}}$ (7). The relation between the AD price matrices $\mathbf{A}_{S\times S}$ and $\overline{\mathbf{A}}_{\overline{S}\times\overline{S}}$ is established by first consolidating (i.e., summing) $\mathbf{A}$'s relevant columns $j \in \overline{j}$ pertaining to a coupled state $\overline{j} \in \overline{\mathcal{S}}$ (separately for every $\overline{j} \in \overline{\mathcal{S}}$) to construct an auxiliary matrix $\mathbf{A}^{+}_{S\times\overline{S}}$,

$$\mathbf{A}^{+}_{:,\overline{j}} \equiv \sum_{j\in\overline{j}} \mathbf{A}_{:,j}, \qquad \overline{j} \in \overline{\mathcal{S}}, \tag{12}$$

where we use $\mathbf{X}_{:,j}$ (and $\mathbf{X}_{j,:}$) to denote the j-th column (and the j-th row) of a generic matrix $\mathbf{X}$. The following characterization of the auxiliary matrix defines the consistency between AD price matrices associated with specifications $\mathcal{S}$ and $\overline{\mathcal{S}}$.

Definition 2 (Consistent AD price matrices) *One-period AD price matrices* $\mathbf{A}_{S\times S}$ *and* $\overline{\mathbf{A}}_{\overline{S}\times\overline{S}}$*, where* $\overline{S} < S$*, are consistent with each other if the auxiliary matrix* $\mathbf{A}^{+}_{S\times\overline{S}}$ (12) *(con-*

structed from matrix $\mathbf{A}$*) has relevant identical rows, which are identical to the corresponding row of matrix* $\overline{\mathbf{A}}$*, i.e.,* $\mathbf{A}^+_{j,:} = \mathbf{A}^+_{k,:} = \overline{\mathbf{A}}_{\bar{j},:}$, $\forall j, k \in \bar{j}$*, separately for each coupled state* $\bar{j} \in \overline{\mathcal{S}}$*.*

This definition generalizes the consistency pattern illustrated in the consolidation (10) of Example 1. This definition posits on a key intuition that consistent recoveries can only arise when AD assets construed by different analysts based on their respective specifications preserve information about the underlying market model. According to Definition 2, when $\mathbf{A}$ and $\overline{\mathbf{A}}$ are consistent, consolidating $\mathbf{A}$ into $\overline{\mathbf{A}}$ involves dropping identical (and redundant) rows of the auxiliary matrix $\mathbf{A}^+$, thus incurring no loss of information. This information-preserving consolidation arises from the consistency of the associated specifications, where all states $j \in \bar{j}$ pertaining to a coupled state $\bar{j}$ are indistinguishable. It is crucial to note that, in general, one-period AD price matrices can only be constructed (implied) from prices, as in (6) and (8), before they can be related. However, when specifications $\overline{\mathcal{S}}$ and $\mathcal{S}$ are consistent, Definition 2 provides an effective procedure to deduce $\overline{\mathbf{A}}_{\overline{S}\times\overline{S}}$ from $\mathbf{A}_{S\times S}$ without any loss of information.

The consistency between AD price matrices implies a strong pairwise correspondence between their eigenspaces of all orders. While the dominant eigenspaces characterize the recovered results (and are thus crucial for the Recovery Theorem), eigenspaces of all orders are essential for analyzing the inconsistencies in these recovered results, as discussed in Section 3.2 below. As a prerequisite for this analysis, the following remark presents the pairwise correspondence between the eigenspaces of consistent AD price matrices.

Remark 1 (Consistent AD eigenspaces) *Let* $\mathbf{A}_{S\times S}$ *and* $\overline{\mathbf{A}}_{\overline{S}\times\overline{S}}$*,* $\overline{S} < S$*, be consistent one-period AD price matrices (Definition 2). Then, (i) every eigenvalue of* $\overline{\mathbf{A}}$ *is an eigenvalue of* $\mathbf{A}$*, (ii) the components of the associated right eigenvectors are pairwise identical for single states, and identical (redundant) for coupled states, and (iii) the components of the associated left eigenvectors are pairwise identical for single states, and consolidated (summed)* *for coupled states. In expressions, these correspondences are,*

$$
\begin{array}{c}
\textit{(i) Eigenvalues' correspondence:} \qquad \overline{\delta}^{(k)} = \delta^{(k)}, \qquad \forall k \in \{1, \ldots, \overline{S}\}, \\
\left\{ \begin{array}{c} \textit{(ii) Right eigenvectors' correspondence:} \\ \textit{Single state } \overline{j}\textit{:} \quad \overline{x}_{\overline{j}}^{(k,R)} = x_j^{(k,R)}, \\ \textit{Coupled state } \overline{j}\textit{:}\ \overline{x}_{\overline{j}}^{(k,R)} = x_j^{(k,R)},\ \forall j \in \overline{j}, \end{array} \right. \quad
\left\{ \begin{array}{c} \textit{(iii) Left eigenvectors' correspondence:} \\ \textit{Single state } \overline{j}\textit{:} \quad \overline{x}_{\overline{j}}^{(k,L)} = x_j^{(k,L)}, \\ \textit{Coupled state } \overline{j}\textit{:}\ \overline{x}_{\overline{j}}^{(k,L)} = \sum_{j \in \overline{j}} x_j^{(k,L)}, \end{array} \right.
\end{array} \tag{13}
$$

where $\overline{x}_{\overline{j}}^{(k,R)}$ (and $\overline{x}_{\overline{j}}^{(k,L)}$) denotes the $\overline{j}$-th component of the k-th right (and left) eigenvector of the consolidated AD price matrix, parallel to the notation for the eigenspaces of the original matrix $\mathbf{A}$. Note that right and left eigenvectors are orthonormal, $\mathbf{X}^L\mathbf{X}^R = \mathbf{X}^R\mathbf{X}^L = \mathbb{1}_{S\times S}$ and $\overline{\mathbf{X}}^L\overline{\mathbf{X}}^R = \overline{\mathbf{X}}^R\overline{\mathbf{X}}^L = \mathbb{1}_{\overline{S}\times\overline{S}}$.

This remark generalizes the consistent right and left eigenvectors of the 3-state model in Example 1 (Equation (11) and Footnote 12) to all orders $k \in \{1, \ldots, \overline{S}\}$ and an arbitrary number of states S. In particular, the leading-order (dominant, $k = 1$) eigenspace corresponds to the consistent premise of the recovery process (Proposition 1): $\overline{\delta}^{(1)} = \delta^{(1)} = \delta$ and $\frac{1}{M_{\overline{j}}} = \overline{x}_{\overline{j}}^{(1,R)} = x_j^{(1,R)} = \frac{1}{M_j}$, and the steady-state probability distribution $\{p_j\}$ satisfies $\sum_{j\in\overline{j}} x_j^{(1,L)} \sim \sum_{j\in\overline{j}} p_j$.[13] For higher-order eigenspaces, the correspondence (13) allows us to effectively deduce eigenvectors of the consolidated $\overline{\mathbf{A}}$ from the corresponding ones of the original $\mathbf{A}$ when these AD price matrices are consistent with each other.

In summary, the same necessary and sufficient condition of Proposition 1, which ensures consistent recovery results (i.e., the dominant eigenspaces of AD price matrices), also assures the consistency of sub-dominant eigenspaces of these matrices. When this condition fails, eigenspaces of all orders become pairwise inconsistent and contribute to the inconsistencies of recovery results as we will explore next.

[13] To see this, recall from the decompositions (3) and the discussion below (5) that the dominant left eigenvector of the AD price matrix and the steady-state probability distribution under the physical measure are related as $x_j^{(1,L)} = M_j p_j$ for all original states $j \in \mathcal{S}$. Since M_j is the same (and equal to $M_{\overline{j}}$) for all original states j belonging to a coupled state $\overline{j}$ in a consistent recovery (Proposition 1), we have $\sum_{j\in\overline{j}} x_j^{(1,L)} = M_{\overline{j}} \sum_{j\in\overline{j}} p_j$.

3.2 General Results

Equipped with characterizations of consistency in state space specifications, AD price matrices, and their eigenspaces, we now analyze recovery inconsistencies by deviating from the premise of Proposition 1. A perturbative setup provides an analytical framework (Section 3.2.1) to gain conceptual insights into inconsistent recoveries for generic specifications of any dimensions (Section 3.2.2). Our subsequent empirical analysis does not assume or impose a perturbative structure on the data.

3.2.1 Perturbative Setup and Analytical Results

The setup: Consider two different analysts implementing the recovery process for the same underlying market model using their respective original and consolidated specifications, which obey a nesting relationship $\mathcal{S} \supset \overline{\mathcal{S}}$. Assume that the current state is a single state, $1 = \overline{1}$. To deviate from the restrictive consistent specification premise (Definition 1), let the underlying marginal utilities (i.e., risk preference) associated with original states belonging to the same coupled state, as well as the time discount factor (i.e., time preference), differ by a perturbative term[14]

$$M_j(\varepsilon) = M_{0\bar{j}} + \varepsilon\, k_j, \quad \forall j \in \bar{j}, \;\; \bar{j} \in \overline{\mathcal{S}}, \qquad \text{and} \qquad \delta(\varepsilon) = \delta_0 + \varepsilon \Delta\delta, \tag{14}$$

where ε is a common small perturbative parameter, $\{k_j\}$ are state-specific real parameters with $|\varepsilon\, k_j| \ll \left|M_{0\bar{j}}\right|$, and $\Delta\delta$ is a state-independent parameter with $|\varepsilon\Delta\delta| \ll |\delta_0|$. Throughout the analysis, the subscript 0 denotes the unperturbed component X_0 in a generic perturbative expansion $X(\varepsilon) = X_0 + \varepsilon\, X$. For simplicity, we assume that the underlying transition probabilities $\{p_{ij}\}$ under the physical measure are consistent, i.e., they satisfy Definition 1.[15]

[14] For original states j_1 and j_2 belonging to different coupled states $\bar{j}_1$ and $\bar{j}_2$, the underlying marginal utilities M_{j_1} and M_{j_2} can differ by an arbitrary (non-perturbative) amount.

[15] Note that a perturbation to either preferences or probabilities will result in a deviation from the consistent specification configuration of Definition 1. While a perturbation to the probabilities can be formulated (similar to (14) of the preference perturbation), it does not change our main finding that recovery

The free parameters $\{k_j\}$ are independent of $\{M_{0\bar{j}}\}$, which help to model the exogeneity (independence) between the perturbative and unperturbed components of the setup. By construction, the unperturbed component $M_{0\bar{j}}$ does not vary with index j, meaning that it is the same for all original states $j \in \bar{j}$ belonging to the same coupled state $\bar{j}$, i.e., a consistent configuration (Definition 1). Below, we summarize two key properties of the perturbative setup (14) designed to quantify and inform our analysis of recovery inconsistencies.

Remark 2 (Perturbative setup properties and comparative analysis) *(i) The absence of the perturbative components, $\varepsilon = 0$ in (14), assures that the two analysts recover consistent results.*

(ii) The recovery inconsistencies therefore require the presence of the perturbative components, $\varepsilon \neq 0$. Fixing (a small) $\varepsilon \neq 0$ and the underlying specification $\mathcal{S}$, our principal comparative analysis studies how the recovery results associated with a subjective specification $\overline{\mathcal{S}}$ differ from the underlying model when $\overline{\mathcal{S}}$ becomes finer and approaches $\mathcal{S}$.

Altogether, the full characteristics $\{\delta(\varepsilon), M_j(\varepsilon), p_{ij}\}$ (14) constitute the underlying (true, data-generating) full model, which all analysts aim to recover. The unperturbed characteristics $\left\{\delta_0, M_{0\bar{j}}, p_{ij}\right\}$ present the leading-order component in our perturbative expansion to demonstrate the inconsistencies in recovering the full underlying model by different analysts.

AD Price Matrices

As all asset prices are generated by the underlying characteristics $\{\delta(\varepsilon), M_i(\varepsilon), p_{ij}\}$, the current τ-period AD asset prices associated with the original (underlying) specification $\mathcal{S}$ satisfy $A_{\tau;1j}(\varepsilon) = \delta(\varepsilon)^\tau p_{\tau;1j} \frac{M_j(\varepsilon)}{M_1(\varepsilon)}$, where 1 denotes the current state, $j \in \mathcal{S}$ the target (payoff) state in τ periods, and $p_{\tau;1j}$ the τ-period transition probability between these states, $p_{\tau;1j} = [\mathbf{P}^\tau]_{1j}$. The first analyst perceives the set of original current prices $\{A_{\tau;1j}(\varepsilon)\}$, $j \in \mathcal{S}$, while

inconsistencies persist as long as analysts' subjective specifications deviate from Definition 1, and it is omitted to simplify the exposition.

the second perceives the set of consolidated prices $\{\overline{A}_{\tau;\bar{1}\bar{j}}(\varepsilon)\}$, $\bar{j} \in \overline{\mathcal{S}}$. Both sets are observable and related to each other by the law of one price, $\overline{A}_{\tau;\bar{1}\bar{j}}(\varepsilon) = \sum_{j\in\bar{j}} A_{\tau;1\,j}(\varepsilon)$, $\forall \bar{j} \in \overline{\mathcal{S}}$.

Under the assumption of a small perturbative parameter ε, the perturbative analysis expresses all quantities of interest as series expansions in powers of ε. For tractability, we retain only the two leading-order terms, i.e., the zero-order term (associated with ε^0) and the linear-order term (associated with ε), of these expressions. In accordance with this linear-expansion approximation for the perturbative setup (14), the observable τ-period AD prices $\{A_{\tau;1j}(\varepsilon)\}$ and $\{\overline{A}_{\tau;\bar{1}\bar{j}}(\varepsilon)\}$ are linear in ε. Stacking these prices into matrices yields the following linear-expansion decompositions[16]

$$\begin{cases} \mathbf{A}_\tau(\varepsilon) = \mathbf{A}_{0\tau} + \varepsilon\,\mathbf{B}_\tau, \\ \mathbf{A}_{\tau+1}(\varepsilon) = \mathbf{A}_{0\tau+1} + \varepsilon\,\mathbf{B}_{\tau+1}, \end{cases} \qquad \begin{cases} \overline{\mathbf{A}}_\tau(\varepsilon) = \overline{\mathbf{A}}_{0\tau} + \varepsilon\,\overline{\mathbf{B}}_\tau, \\ \overline{\mathbf{A}}_{\tau+1}(\varepsilon) = \overline{\mathbf{A}}_{0\tau+1} + \varepsilon\,\overline{\mathbf{B}}_{\tau+1}. \end{cases} \tag{15}$$

The analysts solve for the respective one-period AD price matrices, $\mathbf{A}(\varepsilon) = \mathbf{A}_\tau^{-1}(\varepsilon)\mathbf{A}_{\tau+1}(\varepsilon)$ (6) and $\overline{\mathbf{A}}(\varepsilon) = \overline{\mathbf{A}}_\tau^{-1}(\varepsilon)\overline{\mathbf{A}}_{\tau+1}(\varepsilon)$ (8), and obtain

$$\begin{aligned} \mathbf{A}(\varepsilon) &= \mathbf{A}_0 + \varepsilon\,\mathbf{B}, \quad \text{with} \quad \mathbf{A}_0 = \mathbf{A}_{0\tau}^{-1}\mathbf{A}_{0\tau+1}, \quad \mathbf{B} = \mathbf{A}_{0\tau}^{-1}\left(\mathbf{B}_{\tau+1} - \mathbf{B}_\tau\mathbf{A}_0\right), \\ \overline{\mathbf{A}}(\varepsilon) &= \overline{\mathbf{A}}_0 + \varepsilon\,\overline{\mathbf{B}}, \quad \text{with} \quad \overline{\mathbf{A}}_0 = \overline{\mathbf{A}}_{0\tau}^{-1}\overline{\mathbf{A}}_{0\tau+1}, \quad \overline{\mathbf{B}} = \overline{\mathbf{A}}_{0\tau}^{-1}\left(\overline{\mathbf{B}}_{\tau+1} - \overline{\mathbf{B}}_\tau\overline{\mathbf{A}}_0\right). \end{aligned} \tag{16}$$

These equations yield the (implied) one-period AD price matrices $\mathbf{A}(\varepsilon)$ and $\overline{\mathbf{A}}(\varepsilon)$ in terms of (observable) price matrices $\mathbf{A}_{0\tau}, \mathbf{B}_\tau, \overline{\mathbf{A}}_{0\tau}, \overline{\mathbf{B}}_\tau$ obtained in (15). While these matrices are implied from the inputs of the perturbative setup, the presence of free parameters $\{k_j\}$ in (14) allows us to model and regulate an exogeneity between the perturbative ($\mathbf{B}$, $\overline{\mathbf{B}}$) and unperturbed ($\mathbf{A}_0$, $\overline{\mathbf{A}}_0$) components (see also Footnote 16).

[16]Specifically, substituting the perturbative setup (14) into the τ-period AD price $A_{\tau;1j}(\varepsilon) = \delta(\varepsilon)^\tau p_{\tau;1j}\frac{M_j(\varepsilon)}{M_1(\varepsilon)}$ produces the linear expansion $A_{\tau;1j}(\varepsilon) = \delta^\tau p_{\tau;1j}\frac{M_{0\,\bar{j}}}{M_{0\,\bar{1}}} + \varepsilon\,\delta^\tau p_{\tau;1j}\frac{M_{0\,\bar{j}}}{M_{0\,\bar{1}}}\left(\tau\,\frac{\Delta\delta}{\delta} + \frac{k_j}{M_{0\,\bar{j}}} - \frac{k_1}{M_{0\,\bar{1}}}\right)$. Identifying these expansions with the matrix equation $\mathbf{A}_\tau(\varepsilon) = \mathbf{A}_{0\tau} + \varepsilon\mathbf{B}_\tau$ (15) yields explicit expressions for the components (tj) of the unperturbed and perturbative matrices: $[\mathbf{A}_{0\tau}]_{tj} = \delta^t p_{t;1j}\frac{M_{0\,\bar{j}}}{M_{0\,\bar{1}}}$ and $[\mathbf{B}_\tau]_{tj} = \delta^t p_{t;1j}\frac{M_{0\,\bar{j}}}{M_{0\,\bar{1}}}\left(t\,\frac{\Delta\delta}{\delta} + \frac{k_j}{M_{0\,\bar{j}}} - \frac{k_1}{M_{0\,\bar{1}}}\right)$. Expressions for $\overline{\mathbf{A}}_{0\tau}$ and $\overline{\mathbf{B}}_\tau$ follow similarly from the consolidated τ-period AD prices $\{\overline{A}_{\tau;\bar{1}\bar{j}}(\varepsilon)\}$.

Information Retention in AD Price Matrices

The recovery consistency issue centers on a comparative analysis of the results recovered by different analysts, i.e., whether the dominant eigenspaces of $\mathbf{A}(\varepsilon)$ and $\overline{\mathbf{A}}(\varepsilon)$ are consistent with each other. In light of Remarks 1 and 2, our approach to the recovery consistency issue is to analyze a more fundamental inquiry on whether the construction of the implied matrices $\mathbf{A}(\varepsilon)$ and $\overline{\mathbf{A}}(\varepsilon)$ incurs information loss about the underlying market model, which makes these two matrices, as well as their eigenspaces, inconsistent. Since the unperturbed components $\mathbf{A}_0$ and $\overline{\mathbf{A}}_0$ are consistent (by construction, Remark 2), information loss and recovery inconsistencies can only arise from the perturbative components $\mathbf{B}$ and $\overline{\mathbf{B}}$ and their interaction (coupling) with the unperturbed components. We first characterize the consistency between the perturbative components, before discussing how information is preserved (or lost) in constructing these components from observable prices.

Adopting Definition 2, $\mathbf{B}$ and $\overline{\mathbf{B}}$ are consistent with each other if all rows $j \in \bar{j}$ of the auxiliary matrix $\mathbf{B}^+$ (constructed from $\mathbf{B}$) are identical to the $\bar{j}$-th row of matrix $\overline{\mathbf{B}}$,

$$\mathbf{B}^+_{j,:} = \overline{\mathbf{B}}_{\bar{j},:}, \qquad \forall j \in \bar{j}, \;\; \bar{j} \in \overline{\mathcal{S}}. \tag{17}$$

Practically, when $\mathbf{B}$ and $\overline{\mathbf{B}}$ are consistent, the auxiliary matrix $\mathbf{B}^+$ features relevant identical rows, so they can be dropped without information loss. Due to the exogeneity between the unperturbed and perturbative components of AD price matrices (observed below (16)), depending on the setup's input parameters and specifications $\mathcal{S}$ and $\overline{\mathcal{S}}$, the perturbative components $\mathbf{B}$ and $\overline{\mathbf{B}}$ may be either consistent or inconsistent with each other, while the unperturbed $\mathbf{A}_0$ and $\overline{\mathbf{A}}_0$ remain consistent by construction. When the perturbative components are not constrained by the consistency condition (17), the unperturbed and perturbative components of AD price matrices are exogenous to (independent of) each other in general.

To see how much information is retained in $\overline{\mathbf{B}}$ compared to $\mathbf{B}$, we start with consolidating relevant columns of $\mathbf{B}$ in (16) and obtain[17]

$$\underbrace{\mathbf{A}_{0\tau}}_{S\times S}\underbrace{\mathbf{B}^{+}}_{S\times\overline{S}} = \underbrace{\mathbf{B}^{+}_{\tau+1}}_{S\times\overline{S}} - \underbrace{\mathbf{B}^{+}_{\tau}}_{S\times\overline{S}}\underbrace{\overline{\mathbf{A}}_{0}}_{\overline{S}\times\overline{S}}. \tag{18}$$

There are two exhaustive and mutually exclusive cases concerning this matrix equation.

Case 1: In the special case where the perturbative components are consistent with each other (17), both sides of the matrix equation (18) have $(S-\overline{S})$ redundant rows. These rows can be dropped without loss of information, transforming (18) into a $\overline{S}\times\overline{S}$ matrix equation $\overline{\mathbf{A}}_{0\tau}\,\overline{\mathbf{B}} = \overline{\mathbf{B}}_{\tau+1} - \overline{\mathbf{B}}_{\tau}\,\overline{\mathbf{A}}_{0}$, which yields $\overline{\mathbf{B}}$ in (16).[18] That is, when the perturbative components are consistent, the constructions of $\mathbf{B}$ and $\overline{\mathbf{B}}$ from observable asset prices (16) retain the same amount of information.

Case 2: In the general case where the perturbative components are inconsistent with each other, there are no redundant rows in (18). The second analyst (using the subjective specification $\overline{\mathcal{S}}$ and unaware of the underlying $\mathcal{S}$) only needs $\overline{S}$ recovery equations (8) and drops the $S-\overline{S}$ extra equations (i.e., $S-\overline{S}$ rows of both sides of (18)), even though these rows are not redundant.[19] As a result, in the current case, dropping these rows transforms (18) into a matrix equation that retains less information than (18), $\mathbf{A}_{0\tau-}\,\mathbf{B}^{+} = \overline{\mathbf{B}}_{\tau+1} - \overline{\mathbf{B}}_{\tau}\overline{\mathbf{A}}_{0}$, where the $\overline{S}\times S$ matrix $\mathbf{A}_{0\tau-}$ is constructed from the $S\times S$ matrix $\mathbf{A}_{0\tau}$ by dropping $(S-\overline{S})$ extra rows (Footnote 10).[20] Substituting this equation into the equation determining $\overline{\mathbf{B}}$ in

[17]Summing the relevant columns of both sides of the matrix equation for $\mathbf{B}$ (16) yields $\mathbf{B}^{+} = \mathbf{A}_{0\tau}^{-1}\left(\mathbf{B}^{+}_{\tau+1} - \mathbf{B}_{\tau}\mathbf{A}_{0}^{+}\right)$, or equivalently $\mathbf{A}_{0\tau}\,\mathbf{B}^{+} = \mathbf{B}^{+}_{\tau+1} - \mathbf{B}_{\tau}\mathbf{A}_{0}^{+}$. Since $\mathbf{A}_{0}$ is consistent by construction, $\mathbf{A}_{0}^{+}$ features identical relevant rows (Definition 2), implying $\mathbf{B}_{\tau}\,\mathbf{A}_{0}^{+} = \mathbf{B}^{+}_{\tau}\,\overline{\mathbf{A}}_{0}$ and thus (18).

[18]On the left-hand side (LHS) of (18), when $\mathbf{B}$ and $\overline{\mathbf{B}}$ are consistent, the relevant identical rows of $\mathbf{B}^{+}$ (17) imply an matrix product identity $\mathbf{A}_{0\tau}\,\mathbf{B}^{+} = \mathbf{A}^{+}_{0\tau}\,\overline{\mathbf{B}}$. Since $\mathbf{A}_{0}$ is consistent by construction, $\mathbf{A}^{+}_{0\tau}$ features identical relevant rows that can be dropped without information loss, after which it turns into $\overline{\mathbf{A}}_{0\tau}$ and the LHS of (18) becomes $\overline{\mathbf{A}}_{0\tau}\,\overline{\mathbf{B}}$. On the right-hand side (RHS) of (18), consistent perturbative components (17) imply that the observable matrices $\mathbf{B}^{+}_{\tau+1}$ and $\mathbf{B}^{+}_{\tau}$ also feature identical relevant rows that can be dropped without information loss, reducing them to $\overline{\mathbf{B}}_{\tau+1}$ and $\overline{\mathbf{B}}_{\tau}$. Thus, the RHS of (18) becomes $\overline{\mathbf{B}}_{\tau+1} - \overline{\mathbf{B}}_{\tau}\,\overline{\mathbf{A}}_{0}$.

[19]Employing more price data (tenors) than needed, e.g., in a least-squares recovery implementation, does not weaken the consistency requirement and hence does not alleviate the recovery consistency issue as along as $\mathcal{S}$ and $\overline{\mathcal{S}}$ are inconsistent. Intuitively, as the law of one price that relates asset prices employed under $\mathcal{S}$ and $\overline{\mathcal{S}}$ applies uniformly across various tenors for all price data points, adding tenors does not resolve the issue (Appendix B.2). Section 4 presents empirical evidence for recovery inconsistencies in least-squares implementations.

[20]Note that after dropping $(S-\overline{S})$ extra rows not needed for the recovery in $\overline{\mathcal{S}}$, the auxiliary matrices

(16) yields $\overline{\mathbf{B}} = \overline{\mathbf{A}}_{0\tau}^{-1} \mathbf{A}_{0\tau-} \mathbf{B}^{+}$. As explained earlier, this $\overline{\mathbf{B}}$ retains less information than $\mathbf{B}$ in (16) (which underlies (18)) when the perturbative components are inconsistent. The following proposition summarizes the amount of information retained about the underlying market model in these two cases.

Proposition 2 (Information retention in AD price matrices) *Given the perturbative setup* (14) *and the associated decompositions and constructions* (16) *of the one-period AD price matrices from observable asset prices,*

1/ *when the perturbative components* $\mathbf{B}$ *and* $\overline{\mathbf{B}}$ *are consistent with each other (as quantified by* (17)*), the solutions* (16) *of these matrices retain the same amount of information,*

2/ *when the perturbative components* $\mathbf{B}$ *and* $\overline{\mathbf{B}}$ *are inconsistent, the solution for* $\overline{\mathbf{B}}$ *in* (16) *is equivalent to* $\overline{\mathbf{B}} = \overline{\mathbf{A}}_{0\tau}^{-1} \mathbf{A}_{0\tau-} \mathbf{B}^{+}$*, demonstrating that* $\overline{\mathbf{B}}$ *retains less information than* $\mathbf{B}$ *in this case.*

The key insight of this proposition is that the inconsistency among the perturbative components of AD price matrices is the source of the (relative) information loss among different recovery processes (under $\overline{\mathcal{S}}$ and $\mathcal{S}$). Next we present the recovery results in the perturbative setup before elaborating on the relationship between information loss and recovery inconsistencies in Section 3.2.2.

Recovery Results

Endowed with the original (underlying) specification $\mathcal{S}$, the first analyst solves for the dominant eigenspace of the original one-period AD price matrix (16), $\mathbf{A}(\varepsilon)\mathbf{x}^{(1,R)}(\varepsilon) = \delta^{(1)}(\varepsilon)\mathbf{x}^{(1,R)}(\varepsilon)$,

---

$\mathbf{B}_{\tau+1}^{+}$ and $\mathbf{B}_{\tau}^{+}$ coincide with $\overline{\mathbf{B}}_{\tau+1}$ and $\overline{\mathbf{B}}_{\tau}$, respectively.

which represents the underlying model by construction (Recovery Theorem), and obtains[21]

$$\delta^{(1)}(\varepsilon) = \delta_0^{(1)} + \varepsilon\ \mathbf{x}_0^{(1,L)}\ \mathbf{B}\ \mathbf{x}_0^{(1,R)}, \qquad \mathbf{x}^{(1,R)}(\varepsilon) = \mathbf{x}_0^{(1,R)} + \varepsilon \sum_{k=2}^{S} \frac{\mathbf{x}_0^{(k,L)}\,\mathbf{B}\,\mathbf{x}_0^{(1,R)}}{\delta_0^{(1)} - \delta_0^{(k)}}\,\mathbf{x}_0^{(k,R)}, \tag{19}$$

where $\delta_0^{(k)}$, $\mathbf{x}_0^{(k,R)}$ and $\mathbf{x}_0^{(k,L)}$ are the k-th eigenvalue and right (column) and left (row) eigenvectors of the unperturbed matrix component $\mathbf{A}_0$ in (16). Similarly, the second analyst solves for the dominant consolidated eigenspace $\overline{\mathbf{A}}(\varepsilon)\overline{\mathbf{x}}^{(1,R)}(\varepsilon) = \overline{\delta}^{(1)}(\varepsilon)\overline{\mathbf{x}}^{(1,R)}(\varepsilon)$ in $\overline{\mathcal{S}}$ and obtains

$$\overline{\delta}^{(1)}(\varepsilon) = \overline{\delta}_0^{(1)} + \varepsilon\ \overline{\mathbf{x}}_0^{(1,L)}\ \overline{\mathbf{B}}\ \overline{\mathbf{x}}_0^{(1,R)}, \qquad \overline{\mathbf{x}}^{(1,R)}(\varepsilon) = \overline{\mathbf{x}}_0^{(1,R)} + \varepsilon \sum_{k=2}^{\overline{S}} \frac{\overline{\mathbf{x}}_0^{(k,L)}\,\overline{\mathbf{B}}\,\overline{\mathbf{x}}_0^{(1,R)}}{\overline{\delta}_0^{(1)} - \overline{\delta}_0^{(k)}}\,\overline{\mathbf{x}}_0^{(k,R)}, \tag{20}$$

where $\overline{\delta}_0^{(1)}$, $\overline{\mathbf{x}}_0^{(k,R)}$ and $\overline{\mathbf{x}}_0^{(k,L)}$ are the k-th eigenvalue and right and left eigenvectors of the unperturbed component $\overline{\mathbf{A}}_0$ of the consolidated AD price matrix in (16). Since the unperturbed components are consistent by construction, the recoveries in the limit $\varepsilon = 0$ are consistent. Comparing (19) and (20) with (14) in this limit implies $\delta_0^{(1)} = \overline{\delta}_0^{(1)} = \delta_0$. Hereafter, we substitute the unperturbed consistent dominant eigenvalues $\delta_0^{(1)}$ and $\overline{\delta}_0^{(1)}$ by δ_0.

Unlike the recovered time preferences, the recovered risk preferences represented by the dominant right eigenvectors of the full models are also coupled to all higher-order right and left eigenvectors of the unperturbed components of the models.[22] Perturbative solutions (19) and (20) indicate that these couplings are stronger when the associated (unperturbed) eigenvalues are closer to the dominant (unperturbed) eigenvalues (i.e., smaller distances $|\delta_0^{(k)} - \delta_0|$ and $|\overline{\delta}_0^{(k)} - \delta_0|$). This is because these (unperturbed) eigenvectors have a smaller spectral gap from, and have a stronger influence on the solution of, the dominant eigenvectors of the full models. To illustrate, consider the limit $|\delta_0^{(k)} - \delta_0| \to 0$ (i.e., the dominant and

[21]The the dominant eigenspace (19) follows from the corresponding eigenequation in explicit perturbative expansions, $(\mathbf{A}_0 + \varepsilon\,\mathbf{B})\left(\mathbf{x}_0^{(1,R)} + \varepsilon\,\Delta\mathbf{x}^{(1,R)}\right) = (\delta_0^{(1)} + \varepsilon\,\Delta\delta^{(1)})\left(\mathbf{x}_0^{(1,R)} + \varepsilon\,\Delta\mathbf{x}^{(1,R)}\right)$. We then match terms of same order of ε, multiply to the left of the eigenequation by the left unperturbed eigenvectors $\mathbf{x}_0^{(k,L)}$, $k \in \{2, \ldots, S\}$ of matrix $\mathbf{A}_0$, and employ the orthonormality between these left and right eigenvectors, $\mathbf{X}_0^L \mathbf{X}_0^R = \mathbb{1}_{S\times S}$ to solve for perturbative components $\Delta\delta^{(1)}$ and $\Delta\mathbf{x}^{(1,R)}$.

[22]Full models include both unperturbed and unperturbed components, as discussed below Remark 2.

k-th eigenspaces are degenerate), a (degenerate) perturbative solution can be chosen to be as strongly influenced by the k-th eigenvector as it is by the dominant one. Given the two analysts' recovery results (19) and (20), the following analysis explains their inconsistencies in light of these couplings and the differential information retention across the two recovery systems (Proposition 2).

3.2.2 Recovery Inconsistencies: A Discussion

The perturbative setup is instrumental not only in delivering analytical expressions for the recovery results, but also in delineating separate components that preserve or violate recovery consistency in these results. This delineation informs our current discussion about the source of inconsistencies and their persistence, i.e., how a subjective specification with a larger number of states does not necessarily improve the recovery consistency issue. While the perturbative setup and analysis are for the tractability convenience, their insights into recovery inconsistencies apply as long as the general (non-perturbative) setting is decomposed into consistent and inconsistent components. When these components are non-perturbative, the recovery inconsistencies are plausibly even larger. Our empirical analysis (Section 4) does not assume or impose a perturbative structure on the data.

Inconsistency in the recovered time preferences: To the first order in ε, the divergence in recovered time preferences (19) and (20) is

$$\overline{\delta}^{(1)}(\varepsilon) - \delta^{(1)}(\varepsilon) \;\sim\; \overline{\mathbf{x}}_0^{(1,L)}\, \overline{\mathbf{B}}\, \overline{\mathbf{x}}_0^{(1,R)} \;-\; \mathbf{x}_0^{(1,L)}\, \mathbf{B}\, \mathbf{x}_0^{(1,R)}$$

$$= \underbrace{\left[\overline{\mathbf{x}}_0^{(1,L)}\, \overline{\mathbf{A}}_{0\tau}^{-1} \mathbf{A}_{0\tau-} \;-\; \mathbf{x}_0^{(1,L)}\right]}_{\text{unperturbed \& consistent factor}} \;\times\; \underbrace{\mathbf{B}^{+}\, \overline{\mathbf{x}}_0^{(1,R)}}_{\text{perturbative factor}} \;, \tag{21}$$

where we have employed the general identity $\overline{\mathbf{B}} = \overline{\mathbf{A}}_{0\tau}^{-1}\, \mathbf{A}_{0\tau-}\mathbf{B}^{+}$ (Proposition 2) and the consistency property (13) of the unperturbed right eigenvectors $\mathbf{x}_0^{(1,R)}$ and $\overline{\mathbf{x}}_0^{(1,R)}$, which share relevant identical components, to reduce $\mathbf{B}\,\mathbf{x}_0^{(1,R)}$ to $\mathbf{B}^{+}\,\overline{\mathbf{x}}_0^{(1,R)}$, as in (11). The decomposition

(21) shows that the divergence in recovered time preferences arises from the coupling between a (consistent) unperturbed factor and a perturbative factor. This coupling, as a scalar product of a row and a column vectors on RHS of (21), represents the co-variation (across states) of unperturbed and perturbative factors. As a result, the inconsistency in recovered time preferences on LHS of (21) depends on not only the magnitude of these factors but also on their alignment (i.e., cross-state correlation) in the state space. This co-variation nature in the state space of the divergence in recovered results is key to the persistence of recovery inconsistencies as we explain further below.

In the special case where the perturbative components of the AD price matrices are consistent, terms on the RHS of (21) exactly offset each other,[23] and consequently the recovered time preferences are consistent, $\overline{\delta}^{(1)}(\varepsilon) = \delta^{(1)}(\varepsilon)$. In the general case where the perturbative components $\mathbf{B}$ and $\overline{\mathbf{B}}$ are inconsistent, the two terms on the RHS of (21) do not offset each other, leading to the inconsistency $\overline{\delta}^{(1)}(\varepsilon) \neq \delta^{(1)}(\varepsilon)$. Intuitively, the extra information dropped in the consolidated recovery (under $\overline{\mathcal{S}}$) but needed in the original recovery (under $\mathcal{S}$) is non-redundant (Case 2, Proposition 2). In this case, as $\mathbf{B}^+$ (i.e., perturbative component) is not constrained by the consistency condition (17), the perturbative and unperturbed factors are exogenous to (and independent of) each other in the decomposition (21), precluding the offsetting effect on the RHS of (21) and giving rise to inconsistent recovered discount factors.

In line with Remark 2's comparative analysis, an important question is whether a finer (higher-dimension) subjective specification $\overline{\mathcal{S}}$ delivers a smaller recovery inconsistency. To answer this, recall the co-variation nature between the unperturbed and perturbative components of the divergence (21), as captured by the scalar-product decomposition in the state space. When the perturbative components are inconsistent, the exogeneity between the two factors of the scalar product (21) prevents a universal improvement in their alignment (i.e.,

[23] When $\mathbf{B}$ and $\overline{\mathbf{B}}$ are consistent, (17) implies $\mathbf{A}_{0\tau-}\mathbf{B}^+ = \overline{\mathbf{A}}_{0\tau}\overline{\mathbf{B}}$, and $\mathbf{x}_0^{(1,L)}\mathbf{B}^+ = \overline{\mathbf{x}}_0^{(1,L)}\overline{\mathbf{B}}$ (as in Footnote 12), canceling the RHS of (21).

their correlation in the state space) and, consequently, the consistency, even when a finer subjective specification is employed.

For a fixed underlying model and original specification $\mathcal{S}$, a finer subjective specification $\overline{\mathcal{S}}$ corresponds to a larger number $\overline{S}$ of coupled states, i.e., fewer original states are combined into coupled states in the aggregate. As a result, $\mathbf{B}^+$ is constructed by consolidating (i.e., summing) fewer columns of the original $\mathbf{B}$. This procedure does not necessarily produce a more consistent $\mathbf{B}^+$ (characterized by identical relevant rows $j \in \overline{j}$ within each coupled state $\overline{j} \in \overline{\mathcal{S}}$) because $\mathbf{B}$ is a priori unconstrained (arbitrary). In contrary, it is possible that the resulting $\mathbf{B}^+$ can be even more consistent when $\overline{S}$ decreases. That is, when more columns of $\mathbf{B}$ are consolidated, relevant rows of the resulting $\mathbf{B}^+$ can become more similar. In such cases, the recovery consistency decreases with a finer subjective specification $\overline{\mathcal{S}}$. Appendix B.3.1 elaborates on the quantitative non-monotone relationship between the number $\overline{S}$ of subjective states and the consistency of the associated recovery results by obtaining an explicit expression for the inverse matrix $\overline{\mathbf{A}}_{0\tau}^{-1}$ in the divergence (21) (using Vandermonde matrix).

Inconsistency in the recovered risk preferences: To the first order in ε, the divergence in the recovered risk preferences concerns a comparison of two vectors of dimensions $\overline{S}$ (20) and S (19),

$$\sum_{k=2}^{\overline{S}} \underbrace{\frac{\overline{\mathbf{x}}_0^{(k,L)}\,\overline{\mathbf{B}}\,\overline{\mathbf{x}}_0^{(1,R)}}{\delta_0 - \overline{\delta}_0^{(k)}}}_{\text{loadings } \overline{l}_M^{(k)}} \times \overline{\mathbf{x}}_0^{(k,R)} \quad \text{vs.} \quad \sum_{k=2}^{S} \underbrace{\frac{\mathbf{x}_0^{(k,L)}\,\mathbf{B}\,\mathbf{x}_0^{(1,R)}}{\delta_0 - \delta_0^{(k)}}}_{\text{loadings } l_M^{(k)}} \times \mathbf{x}_0^{(k,R)}, \tag{22}$$

where $\overline{l}_M^{(k)}$ and $l_M^{(k)}$ are the loadings of the recovered risk preferences $\overline{\mathbf{x}}^{(1,R)}(\varepsilon)$ and $\mathbf{x}^{(1,R)}(\varepsilon)$ respectively on the k-th (unperturbed) eigenvectors $\overline{\mathbf{x}}_0^{(k,R)}$ and $\mathbf{x}_0^{(k,R)}$. For each $k \in \{1, \ldots, \overline{S}\}$, since the unperturbed eigenvectors $\overline{\mathbf{x}}_0^{(k,R)}$ and $\mathbf{x}_0^{(k,R)}$ are consistent, their relevant corresponding components are identical (Remark 1). Hence, these loadings are our principal quantities of interest in the analysis of the divergence (22).

First, we consider the paired loadings, i.e., $\overline{l}_M^{(k)}$ and $l_M^{(k)}$ for common $k \in \{2, \ldots, \overline{S}\}$. Since the k-th corresponding unperturbed eigenvalues are identical, $\overline{\delta}_0^{(k)} = \delta_0^{(k)}$ (Remark 1), the divergence of the k-th corresponding loadings can be decomposed as

$$\bar{l}_M^{(k)} - l_M^{(k)} = \frac{1}{\delta_0 - \delta_0^{(k)}} \times \left[\overline{\mathbf{x}}_0^{(k,L)}\, \overline{\mathbf{B}}\, \overline{\mathbf{x}}_0^{(1,R)} - \mathbf{x}_0^{(k,L)}\, \mathbf{B}\, \mathbf{x}_0^{(1,R)}\right]$$

$$= \underbrace{\frac{1}{\delta_0 - \delta_0^{(k)}}}_{\text{distance factor}} \times \underbrace{\left[\overline{\mathbf{x}}_0^{(k,L)}\, \overline{\mathbf{A}}_{0\tau}^{-1} \mathbf{A}_{0\tau-} - \mathbf{x}_0^{(k,L)}\right]}_{\text{unperturbed \& consistent factor}} \times \underbrace{\mathbf{B}^+\, \overline{\mathbf{x}}_0^{(1,R)}}_{\text{perturbative factor}}, \quad k \in \{2, \ldots, \overline{S}\}, \qquad (23)$$

where the last equality arises in an identical fashion that delivered the divergence in time preferences (21). The divergence in the k-th corresponding loadings (23) similarly is a scalar product of an unperturbed (consistent) and a perturbative vector-valued factors, but also scaled by the spectral gap between the k-th and dominant eigenvalues of the (unperturbed) AD price matrices as discussed below (20). In the special case of consistent perturbative components, the recovery implementations in the original and consolidated specifications incur no loss of information, which preserve the consistency between the corresponding loadings, $\bar{l}_M^{(k)} = l_M^{(k)}$, independent of distance factors $\frac{1}{\left|\delta_0^{(k)} - \delta_0\right|}$, for all $k \in \{2, \ldots, \overline{S}\}$. Appendix B.3.2 presents a further quantitative analysis of AD price matrix's spectral gap.

In the general case of inconsistent perturbative components, the intuition for the divergence in loadings (23) and recovered time preferences (21) is similar, i.e., $\mathbf{B}$ and $\overline{\mathbf{B}}$ retain different amounts of price information, leading to inconsistent loadings $\bar{l}_M^{(k)} \neq l_M^{(k)}$. However, unlike recovered time preferences, the recovered risk preferences (22) consist of multiple and heterogeneous terms of loadings, highlighting the role of AD price matrices' spectral gaps $\left\{\left|\delta_0^{(k)} - \delta_0\right|\right\}$, $k \in \{2, \ldots, \overline{S}\}$, in the robustness of the recovery inconsistencies (Appendix B.3.2). A finer subjective specification $\overline{\mathcal{S}}$ does not warrant an improvement for the inconsistencies because $\overline{\mathbf{B}}$ is not necessarily more consistent with $\mathbf{B}$ when $\overline{S}$ increases, for the same reasons underlying the inconsistency persistence of the recovered time preferences (21) discussed earlier. As long as AD price matrices are inconsistent (Definition 2) and have a dense spectrum (large scaling factors $\frac{1}{\left|\delta_0^{(k)} - \delta_0\right|}$, $k \in \{2, \ldots, \overline{S}\}$), the inconsistencies in the recovered risk preference can remain significant as $\overline{\mathcal{S}}$ approaches $\mathcal{S}$.

Next, we consider the unpaired loadings, i.e., the extra loadings in the original recovery $\{l_M^{(k)}\}$, with $k \in \{\overline{S}+1, \ldots, S\}$, that have no counterpart $\bar{l}_M^{(k)}$ in the consolidated recovery (22) and (23). In the special case of consistent perturbative components, there is a decoupling between unpaired right and left eigenvectors of the unperturbed AD price matrix $\mathbf{A}_0$ as follows[24]

$$\mathbf{x}_0^{(j,L)}\,\mathbf{B}\,\mathbf{x}_0^{(k,R)} = \mathbf{x}_0^{(j,L)}\,\mathbf{B}^+\,\overline{\mathbf{x}}_0^{(k,R)} = 0, \qquad \forall j \in \{\overline{S}+1, \ldots, S\},\;\; k \in \{1, \ldots, \overline{S}\}. \tag{24}$$

Intuitively, when perturbative components are also consistent, the extra (unpaired) eigenvectors $\{\mathbf{x}_0^{(j,L)}\}$, $j \in \{\overline{S}+1, \ldots, S\}$, do not add any relevant information to what already retained in $\mathbf{B}^+$, reflected in the orthogonality $\mathbf{x}_0^{(j,L)}\,\mathbf{B}^+ = \mathbf{0}_{1\times\overline{S}}$ (Footnote 24), leading to (24). As a result of the decoupling (24), all unpaired terms contributing to the recovered risk preference under $\mathcal{S}$ are zero, which equalizes the two sums in (22) and assures consistent recovered risk preferences. This delicate cancellation of the unpaired terms in the special case of consistent perturbative components indicates the prevalence of recovery inconsistencies in the general case of inconsistent perturbative components. The presence of non-vanishing unpaired terms broadens the divergence (22) between the recovered risk preferences, above and beyond the inconsistency generated by the paired terms (considered above) in this general case.

In summary, the qualitative findings on recovery inconsistencies are as follows. Inconsistencies in recovered time and risk preferences arise because different analysts adopt different and inconsistent subjective specifications in general, leading to different information retained

[24] The first equality in (24) arises from the fact that consistent right eigenvectors $\mathbf{x}_0^{(k,R)}$ and $\overline{\mathbf{x}}_0^{(k,R)}$ have relevant identical components (13), and therefore $\mathbf{B}\,\mathbf{x}_0^{(k,R)} = \mathbf{B}^+\,\overline{\mathbf{x}}_0^{(k,R)}$, $\forall k \in \{1, \ldots, \overline{S}\}$. For the second equality, note that columns of a consistent auxiliary matrix $\mathbf{B}^+$ (17) and consistent right eigenvectors $\mathbf{x}_0^{(k,R)}$ (Remark 1) share the property that they all have relevant identical components. As a result, $\overline{S}$ columns of $\mathbf{B}^+$ are spanned by right eigenvectors $\mathbf{x}_0^{(k,R)}$, $k \in \{1, \ldots, \overline{S}\}$. The orthogonality between right and left eigenvectors then implies the orthogonality between columns of $\mathbf{B}^+$ and $(S - \overline{S})$ unpaired (extra) left eigenvectors $\mathbf{x}_0^{(j,L)}$, $j \in \{\overline{S}+1, \ldots, S\}$, or $\mathbf{x}_0^{(j,L)}\,\mathbf{B}^+ = \mathbf{0}_{1\times\overline{S}}$.

in their (implied) one-period AD price matrices. This loss of information results in inconsistent dominant and sub-dominant eigenspaces of these AD price matrices, which in turn characterize inconsistent recovery results obtained by different analysts. Our perturbative analysis demonstrates that finer subjective specifications do not necessarily improve recovery consistency, as they are not necessarily more consistent with the unobserved underlying specification. Appendix B.3 provides further quantitative supports for these findings. Section 4.3 below introduces several measures to quantify violations of recovery consistency in the data and documents an increase in recovery inconsistencies with these measures.

Continuous Setting

We conclude the current section with a brief discussion of a parallel continuous setting, which is instructive to the above finding on how a finer subjective specification does not necessarily improve the recovery consistency issue in the discrete setting. Let the continuous state space specification be characterized by a stochastic process representing the underlying state variable y_t with the following risk-neutral dynamics

$$\frac{dy_t}{y_t} = \mu_y^Q dt + \sigma_y dB_t^Q, \tag{25}$$

where B_t^Q is a standard Brownian motion under the risk-neutral measure and μ_y^Q and σ_y are processes adapted to the natural filtration generated by B_t^Q. Let $V(y_t)$ denote the state-contingent price process of a generic traded asset. The risk-neutral pricing of this asset, $V(y_t) = E_t^Q\left[e^{-r_t dt}V(y_{t+dt})\right]$, together with an application of the Itô's lemma on the state dynamics (25) implies that

$$V(y_t) = V(y_t) + E_t^Q\left[e^{-r_t dt}V(y_{t+dt}) - V(y_t)\right] = \left[\mathbb{1} + dt(-r_t + \mathcal{D}^Q)\right]V(y_t), \tag{26}$$

with the infinitesimal generator $\mathcal{D}^Q = \mu_y^Q\, y_t \frac{\partial}{\partial y_t} + \frac{1}{2}\,\sigma_y^2\, y_t^2 \frac{\partial^2}{\partial y_t^2}$. The comparison of the pricing equation (26) with the one using state prices, $V_i = \sum_k A_{ik} V_k$, identifies a mapping between the dt-period AD price matrix and the infinitesimal generator, $\mathbf{A} \longleftrightarrow \mathbb{1} + dt(-r_t + \mathcal{D}^Q)$ in a discretized setting, or

$$\sum_{y_k} A(y_i, y_k) V(y_k) = V(y_i) + dt \left[-r_i + \mu_y^Q \, y_i \, \frac{\partial}{\partial y_i} + \frac{1}{2} \, \sigma_y^2 \, y_i^2 \, \frac{\partial^2}{\partial y_i^2} \right] V(y_i). \tag{27}$$

While an explicit construction for $A_{ik} \equiv A(y_i, y_k)$ can be obtained by applying a simple finite-difference scheme on the right-hand side of the mapping (27), not all finite-difference schemes deliver a consistent construction for the AD price matrix.[25] It is important to note that this inconsistency originates from the underlying state distribution's inputs on the determination of the infinitesimal operator. Indeed, the infinitesimal operator $\mathcal{D}^Q$ arises from the conditional expectation of the infinitesimal expansion

$$\underbrace{\frac{1}{dt} E_t^Q \left[V(y_{t+dt}) - V(y_t) \right]}_{= \mathcal{D}^Q V(y_t)} = \frac{\partial V(y_t)}{\partial y_t} \times \underbrace{\frac{1}{dt} E_t^Q \left[dy_t \right]}_{= \mu_y^Q \, y_t} + \frac{1}{2} \frac{\partial^2 V(y_t)}{\partial y_t^2} \times \underbrace{\frac{1}{dt} E_t^Q \left[(dy_t)^2 \right]}_{= \sigma_y^2 \, y_t^2}, \tag{28}$$

where the last equality is a convergence in the probability limit in accordance with the normally distributed of the state variable $\frac{dy_t}{y_t} \in \mathcal{N}\left(\mu_y^Q dt, \sigma_y^2 dt \right)$ (25). As a result, an ad-hoc finite-difference scheme of the differentials $\frac{\partial}{\partial y_t} V(y_t)$ and $\frac{\partial^2}{\partial y_t^2} V(y_t)$ that does not take into account the underlying state variable distribution may entail an inconsistent construction of the AD price matrix and inconsistent associated recovery results. In particular, a finer finite-difference scheme (i.e., a smaller step size dy of the state space grid) may generate negative (and hence, inconsistent) AD prices (see Footnote 25). This feature of the continuous setting mirrors the recovery consistency issue in the discrete setting analyzed earlier, in which a finer subjective specification may not mitigate the divergence between the analyst's recovery

[25] To illustrate, consider the simple finite-difference scheme $\frac{\partial}{\partial y_i} V(y_i) = \frac{V(y_{i+1}) - V(y_{i-1})}{2dy}$ and $\frac{\partial^2}{\partial y_i^2} V(y_i) = \frac{V(y_{i+1}) + V(y_{i-1}) - 2V(y_i)}{(dy)^2}$ for the mapping (27). This scheme delivers $A_{ii-1} \equiv A(y_i, y_i - dy) = dt \left[-\frac{1}{2dy} \, \mu_y^Q \, y_i + \frac{1}{2(dy)^2} \, \sigma_y^2 \, y_i^2 \right]$ and $A_{ii} \equiv A(y_i, y_i) = 1 - dt \left[r_i + \frac{1}{(dy)^2} \, \sigma_y^2 \, y_i^2 \right]$. The AD price A_{ii-1} is negative (and hence, inconsistent) when the finite-difference scheme is such that $dy > \frac{\sigma_y^2 \, y_i}{\mu_y^Q} > 0$. Similarly, the AD price A_{ii} is negative (and hence, inconsistent) when $0 < dy < \sigma_y \, y_i$.

results and the underlying characteristics as long as the subjective specification remains inconsistent.

4 Empirical Analysis

This section presents an empirical analysis of the recovery consistency issue. Section 4.1 describes sources and properties of data. Section 4.2 describes the neural network and regularization methodologies of a recent literature that we employ in the recovery implementation step. Section 4.3 formulates various theory-implied recovery consistency measures, presents the time series of recovery results under different specifications, and demonstrates their inconsistencies using these consistency measures. Appendix A reports the robustness of these empirical results and analysis for alternative sample periods and specifications.

Altogether, employing the related literature's advanced methodologies addressing large but sparse and noisy option price data, our empirical analysis demonstrates a significant and robust recovery consistency issue in different sample periods and across different specifications (i.e., models). While we quantify inconsistencies as irreconcilable differences between the recovery results of two models, it is important to emphasize that our empirical analysis does not assume either specification to be the data-generating (true but unobserved) model.

4.1 Data

For our subsequent recovery consistency analysis, we first closely follow the empirical methodology of Ludwig (2015) and Audrino et al. (2021) who employ neural network techniques to obtain a robust estimate of the implied volatility surface and state prices. To validate our replication of this methodology, we repeat our data collection and entire analysis in Appendix A to conform with their data and original sample periods (referred to as the benchmarks hereafter).

We employ out-of-the-money (OTM) call and put options on S&P 500 for each Wednesday between January 5, 2000 and August 30, 2023 for the analysis in the main text, and use the data up to December 26, 2012 in Appendix A for validation.[26] We obtain daily closing option and spot prices, interest rates, and index dividend yields from OptionMetrics. We keep the options with the average of the (best) bid and ask prices above \$0.50. We employ the convention in which moneyness $m \equiv \frac{K}{X}$ is the ratio of strike (K) to spot price (X) and tenor τ indicates days to maturity. We restrict the moneyness m's and tenor τ's domains to $m \in [0.4, 1.6]$ and $\tau \in [20, 730]$.

In general, OptionMetrics does not provide interest rates (r) for all tenors for a given date. Therefore, for each date, we linearly interpolate and extrapolate the data so that we obtain the full term structure of interest rates. Following Aït-Sahalia and Lo (1998), for each date and tenor in the data, we compute the implied forward prices (F) from close to at-the-money (ATM) call and put pairs, i.e., $m \in [0.99, 1.01]$, using the put-call parity. We then derive the implied dividend yield (d) from the spot-forward parity $F = Xe^{(r-d)\tau}$. However, the ATM option pairs might not be available for all combinations of dates and tenors in the data. In such a case, we supplement our data by using the S&P 500 dividend yield from OptionMetrics, and we apply the same interpolation and extrapolation procedure for the dividend yield data. Given the dividend yield, we are able to transform the OTM put options into in-the-money (ITM) calls. Hence, our empirical implementation will focus solely on the call options.

Lastly, following literature, we require that the options satisfy the general price bounds,

$$Xe^{-d\tau} \geq C \geq \max\{0, (F - K)e^{-r\tau}\}, \tag{29}$$

[26] We use the previous trading day if there was no trading on a Wednesday.

and the restrictions on vertical and butterfly spreads

$$- e^{-r\tau} \leq \frac{\partial C}{\partial K} \leq 0 \quad \text{and} \quad \frac{\partial^2 C}{\partial K^2} \geq 0, \tag{30}$$

where C denotes call option price. We exclude options that violate these conditions. Our final sample consists of 1,326,696 call options, and the summary statistics of average implied volatilities and prices for different moneyness and tenors is given in Table 1.

Table 1: Summary statistics of call options

m	DITM < 0.90	ITM $[0.90, 0.99]$	ATM $(0.99, 1.01)$	OTM $[1.02, 1.10]$	DOTM > 1.10
Panel A: Maturity $<$ 180 days					
IV (%)	33.22	20.17	15.60	14.26	18.64
Price	634.08	189.42	72.67	30.35	8.86
N	279,836	245,371	70,162	215,747	54,798
Panel B: Maturity 180 — 365 days					
IV (%)	31.29	21.14	18.95	16.90	16.08
Price	920.65	301.12	192.05	112.56	23.85
N	143,439	39,841	8,984	39,797	62,007
Panel C: Maturity $>$ 365 days					
IV (%)	28.54	20.88	19.31	17.84	15.86
Price	920.15	344.86	245.54	170.05	44.50
N	81,511	20,421	4,574	20,050	40,158

Notes: This table reports the average implied volatilities (IV) and prices of the call options on S&P 500 in our sample for different categories of moneyness m and maturities. Moneyness is defined as the ratio of strike to spot price. Our sample contains option data for every Wednesday from January 5, 2000 to August 30, 2023.

Table 1 broadly reproduces the features of the benchmark data employed by Audrino et al. (2021) for the extended sample period to August 2023 (also see Table A.1 in Appendix A for the sample period matching theirs). The implied volatilities, which are key to our subsequent empirical analysis, are also in line with the summary statistics in Ludwig (2015), whose use of the neural network methodology in obtaining the IV surface estimate is adopted by both Audrino et al. (2021) and the current paper. These findings provide a necessary validation for the data processing step before we implement the recovery.

4.2 Recovery Implementation: Outline and Methodologies

The empirical procedure for the recovery consistency analysis is outlined below. First, for each date in our sample, we obtain an implied volatility (IV) surface that is smooth, arbitrage-free, and parametrized by a fine grid of 601 moneyness states. We refer to this fine grid as the original specification $\mathcal{S}$ or the full model, which is associated with the first analyst in the thought experiment (Step 1). Next, we implement a recovery process to obtain the time and risk preferences and probability distribution under the physical measure associated with the fine grid of the IV surface (Step 2). Finally, we use the law of one price to construct observable state prices associated with a consolidated grid of 150 moneyness states (referred to as the consolidated specification $\overline{\mathcal{S}}$, or the consolidated model associated with the second analyst) and implement another recovery process for this consolidated model (Step 3). A comparative analysis of recovery results under $\mathcal{S}$ and $\overline{\mathcal{S}}$ is key to our paper's demonstration of recovery inconsistencies (Section 4.3). Our empirical results are robust to the choices of $\mathcal{S}$ and $\overline{\mathcal{S}}$.[27]

Step 1: Implied Volatility Surface and State Prices

To determine the τ-period AD price matrix (also referred to as the state prices hereafter) $\mathbf{A}_\tau$ (6) via Breeden and Litzenberger (1978)'s methodology, we adopt the neural network approach of the literature to generate the implied volatility (IV) surface for moneyness $m \in [0.4, 1.6]$ and tenor $\tau \in [20, 730]$. This approach is flexible and robust and also results in a smooth, arbitrage-free IV surface.

The key idea behind this approach is to model the implied total variance $\nu(m, \tau)$ using

[27] For robustness check, Appendix A.3.2 presents the recovery inconsistency results associated with 150- and 1201-state specifications employed in the literature.

a two-layer neural network

$$\nu(m,\tau) \equiv \sigma^2(m,\tau)\tau = \beta_0 + \sum_{i=1}^{N} \beta_i f(\alpha_{0i} + \alpha_{1i} m + \alpha_{2i}\sqrt{\tau}), \tag{31}$$

where $\sigma(m,\tau)$ is the implied volatility, $f(x) = \frac{1}{1+e^{-x}}$ is the sigmoid activation function, and N is the number of neurons (or neural nodes) in the hidden layers.[28]

Once we obtain 15 no-arbitrage solutions of IV surface, we take the average of the 5 solutions with lowest residual sum of squares. Finally, we compute the $T \times S$ matrix of state prices $\mathbf{A}_\tau$ using numerical differentiation, where T is the number of tenors of the observable AD prices. We provide further details of this approach in Appendix A.2.1.

Step 2: Recovery of the Full Model

We start with recovering the full one-period (monthly) AD transition matrix $\mathbf{A}$ of $S = 601$ states (also referred to as the underlying model in our thought experiment's terminology). As pointed out in the literature, directly computing $\mathbf{A}$ via (6) can lead to unstable solutions because $\mathbf{A}_\tau$ is often ill-conditioned. Therefore, we solve $\mathbf{A}$ using ridge regression with an endogenous L^2 penalty parameter determined by minimizing the generalized Kullback-Leibler divergence between the state prices $\mathbf{A}_\tau$ implied from the data and those generated by the Markov transition model (6) (see details in Appendix A.2.2). After obtaining the state price transition matrix (i.e., one-period AD price matrix) $\mathbf{A}$, we solve and identify its dominant eigenspace $(\delta^{(1)}, \mathbf{x}^{(1,R)})$ with the recovered time and risk preferences and subsequently determine the recovered transition probabilities under the physical measure (4).[29]

[28] For each neuron i, α_{0i} and $(\alpha_{1i}, \alpha_{2i})$ represent the bias and weights of input data, respectively, and β_i denotes the weight of the output. β_0 is the bias in the neural network.

[29] The τ-period physical and risk-neutral transition probabilities from state i to state j respectively are $p_{t,t+\tau}(i,j) = \delta^{-1} A_{\tau;ij}\, x_i^{(1,R)}$ and $q_{t,t+\tau}(i,j) = e^{r\tau} A_{\tau;ij}$, where $x_i^{(1,R)}$ denotes the i-th entry of vector $\mathbf{x}^{(1,R)}$, and $A_{\tau;ij}$ denotes the (i,j) entry of matrix $\mathbf{A}_\tau$.

Step 3: Recovery of the Consolidated Model

To empirically assess the impact of state space specification on recovery results, we consider a second (and separate) recovery implementation based on the specification $\overline{\mathcal{S}}$ (referred to as the consolidated model in our thought experiment's terminology). For clarity of the empirical analysis and exposition, we consider a nesting structure $\overline{\mathcal{S}} \subset \mathcal{S}$ (7) of the two specifications. This aims to model the feature that some underlying states of $\mathcal{S}$ associated with the first analyst are inadvertently combined (consolidated) into a state of the second analyst's subjective coarser specification $\overline{\mathcal{S}}$. While the AD assets initiated in the current state and employed in the recovery process are perceived by analysts in accordance with their subjective specifications, these assets are traded. Therefore, their prices are necessarily related by the law of one price (i.e., the consolidation process) to prevent arbitrage opportunities. From the original $S = 601$ states and the observable current asset prices in the full model, we employ this consolidation process to obtain the observable current asset prices in the consolidated model $\overline{\mathcal{S}}$ of $\overline{S} = 150$ states. We then estimate the one-period (monthly) associated AD transition matrix $\overline{\mathbf{A}}$ of 150 consolidated states (using the ridge regression with an endogenous L^2 regularization as in Step 2 above). Finally, we recover the probability transition matrix $\overline{\mathbf{P}}$ and marginal utilities $\overline{\mathbf{M}}$ in the consolidated model $\overline{\mathcal{S}}$.

4.3 Empirical Results

We first report the summary statistics for the recovered full, 601-state model. We then demonstrate recovery inconsistencies by comparing the full model's recovered marginal utilities and transition probabilities with those of the consolidated model. We further analyze how recovery inconsistencies vary with theory-implied measures quantifying the violation of the recovery consistency condition in the data. The empirical analysis in the main text concerns the sample period from January 5, 2000 to August 30, 2023. Appendix A.3.1 presents robustness results using data up to December 26, 2012 (the benchmark sample period) for a validation of the our recovery implementation.

Recovery Results

Following procedures detailed in the previous section to estimate the implied volatility surface and implement the recovery, we obtain the risk-neutral and physical (recovered) probability distributions for one-month ahead transitions. For each date in our sample, we compute the one-month ahead moments (mean, volatility, skewness, and kurtosis) of returns based on the values of S&P 500 in all states, using their associated (risk-neutral and physical) probabilities just obtained. This produces a time series for each of these four cross-sectional return moments under the risk-neutral and physical measures. Based on these time series, Table 2 reports the summary statistics of the first four moments of the risk-neutral and recovered 30-day-to-maturity cross-sectional returns. Both risk-neutral and recovered return distributions feature negative skewness and excess kurtosis, signifying the presence of tail events in the stock index returns. The median of the mean return under the risk-neutral measure is negative, which is driven by the fact that the risk-free rate is lower than the dividend yield during periods of the low interest rate regime (Audrino et al. (2021)).

Recovery Inconsistencies in Risk Preferences

Motivated and guided by our previous conceptual analysis of the recovery consistency issue, we empirically identify the premises where the recovery consistency condition is more likely to be violated. We then document and verify that recovery results are indeed inconsistent in these premises. One such premise is where the marginal utility attains its extrema (lowest or highest values). For each date in our sample, we identify and record the value of the minimum recovered marginal utility $\overline{M}_{\bar{\jmath}_{\min}}$ of the consolidated model, with $\bar{\jmath}_{\min}$ denoting the associated consolidated state in which the minimum recovered marginal utility takes place. Then, among the original states of the full model that correspond (i.e., belong) to this consolidated state, we identify and record the values of the minimum, maximum and median

Table 2: Summary statistics of risk-neutral and recovered moments

	Median	Std dev	Min	25th	75th	Max
Panel A: Risk-neutral moments						
Mean (%)	−2.02	4.04	−41.95	−3.71	−0.01	5.07
Volatility (%)	19.34	8.36	9.65	15.26	24.55	79.99
Skewness	−1.60	0.75	−5.16	−2.20	−1.17	−0.01
Kurtosis	10.12	7.94	2.97	6.75	15.52	56.10
Panel B: Recovered moments						
Mean (%)	10.63	5.08	−26.28	7.62	14.24	37.44
Volatility (%)	14.88	7.14	6.39	11.89	19.63	66.99
Skewness	−1.11	0.52	−5.76	−1.47	−0.85	0.00
Kurtosis	6.70	3.74	3.00	5.23	8.97	55.60

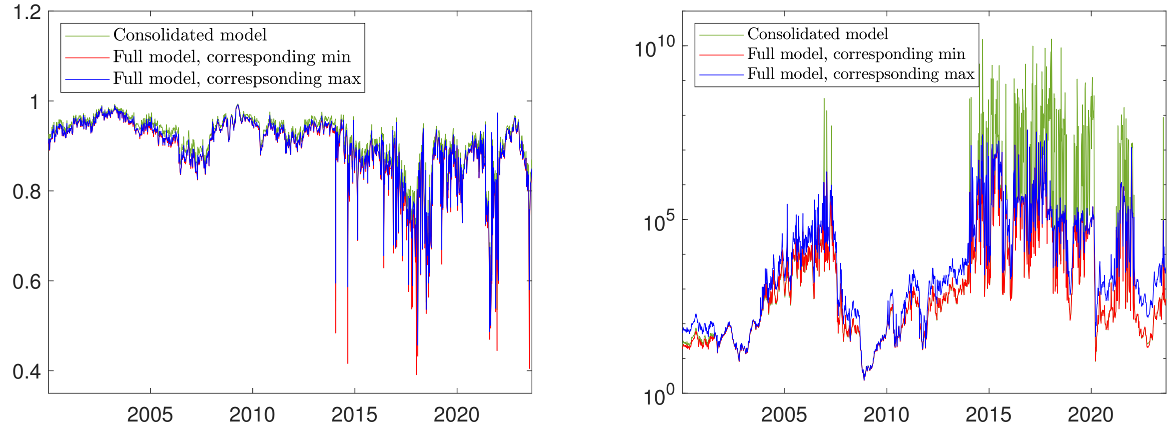
Notes: This table reports the summary statistics of the risk-neutral and recovered moments of 30-day-to-maturity cross-sectional returns on S&P 500 for the 601-state full model. Returns are computed by considering all possible values of S&P 500 on the grid in the next period. Mean and volatility are annualized and reported as percentages. All moments are unconditional over the period of January 5, 2000 to August 30, 2023.

marginal utilities ($M_{j_{\min}}$, $M_{j_{\max}}$ and $M_{j_{\text{med}}}$, respectively), where $j_{\min}, j_{\max}, j_{\text{med}} \in \bar{j}_{\min}$.[30]

Panel A of Figure 1 plots the time series of the recovered marginal utility $\overline{M}_{\bar{j}_{\min}}$ (green line) of the consolidated model and the corresponding recovered marginal utilities $M_{j_{\min}}$ (red line) and $M_{j_{\max}}$ (blue line) of the full model. The plot shows a difference between $M_{j_{\min}}$ and $M_{j_{\max}}$ that persists in most dates of the sample period (from January 2000 to August 2023). This difference becomes significantly more pronounced in the recent period (from 2014 onward), coinciding with the period when $\overline{M}_{\bar{j}_{\min}}$ also differs more significantly from $M_{j_{\min}}$ and $M_{j_{\max}}$.[31] Contrasting this empirical pattern with Proposition 1's necessary and sufficient condition for consistent recoveries, which posits on equal marginal utilities M_j for all original states j belonging to a consolidated state $\bar{j}$, indicates that this condition is violated for states around the minimum recovered marginal utility in the consolidated

[30] Note that $M_{j_{\min}}, M_{j_{\max}}$ or $M_{j_{\text{med}}}$ do not necessarily represent the global minimum, maximum or median marginal utilities across all states of the full model. Furthermore, these marginal utilities will likely differ for different consolidated states.

[31] This observation informs, and is supported by, a more formal statistical correlation analysis of the level of recovery inconsistencies and measures quantifying the violation of the recovery consistent condition in the data (Tables 3 and A.3).



(A) Minimum marginal utilities of the consolidated model

(B) Maximum marginal utilities of the consolidated model

Figure 1: Comparison of marginal utilities of the consolidated and full models

Notes: This figure plots the time series of the recovered minimum ($\overline{M}_{\bar{\jmath}_{\min}}$) and maximum ($\overline{M}_{\bar{\jmath}_{\max}}$) marginal utilities of the (150-state) consolidated model, along with the minimum ($M_{\jmath_{\min}}$) and maximum ($M_{\jmath_{\min}}$) marginal utilities among the original states of the (601-state) full model that correspond to the minimum ($\jmath_{\min} \in \bar{\jmath}_{\min}$) and maximum ($\jmath_{\max} \in \bar{\jmath}_{\max}$) marginal utilities states respectively in the consolidated model. Current state's marginal utility is normalized to one. The sample period is every Wednesday from January 5, 2000 to August 30, 2023.

model (and more significantly violated for the period from 2014 onward). We verify this indication by comparing this recovered marginal utility $\overline{M}_{\bar{\jmath}_{\min}}$ of the consolidated model with all corresponding recovered marginal utilities $\{M_\jmath\}$, $\jmath \in \bar{\jmath}_{\min}$ of the full model. Not only $\overline{M}_{\bar{\jmath}_{\min}}$ do not coincide with either of the minimum $M_{\jmath_{\min}}$ and maximum $M_{\jmath_{\max}}$ (and median $M_{\jmath_{\text{med}}}$, Figure A.2 in Appendix A.3) of the corresponding marginal utilities in the full model, but also $\overline{M}_{\bar{\jmath}_{\min}}$ is outside of the range $[M_{\jmath_{\min}}, M_{\jmath_{\max}}]$ for most of the dates in the sample. That is, the second analyst recovering $\overline{M}_{\bar{\jmath}_{\min}}$ for the consolidated state $\bar{\jmath}_{\min}$ cannot reconcile the result with the marginal utility range $[M_{\jmath_{\min}}, M_{\jmath_{\max}}]$ recovered by the first analyst for all original states $\jmath$ belonging to the consolidated state $\bar{\jmath}_{\min}$, i.e., inconsistent recovered risk preferences.

We repeat this empirical analysis of recovery inconsistencies for the premise of the maximum recovered marginal utility $\overline{M}_{\bar{\jmath}_{\max}}$ of the consolidated model. Panel B of Figure 1 plots the time series of the recovered marginal utility $\overline{M}_{\bar{\jmath}_{\max}}$ of the consolidated model and the corresponding range of the recovered marginal utilities of the full model. This plot exhibits similar patterns, but with significantly larger magnitudes, than those observed in Panel A. These results indicate significant and persistent inconsistencies between the recovered $\overline{M}_{\bar{j}_{\max}}$ and the recovered marginal utilities at the corresponding original states. They also indicate that recovery inconsistencies are significantly larger around consolidated states of highest marginal utilities (i.e., adverse states) than around those of lowest marginal utilities (i.e., good states). Intuitively, given that adverse states of the underlying market tend to be rarer, they plausibly are more elusive to the recovery process than other states. Further empirical evidence for inconsistencies in the recovered risk preferences for the benchmark period is presented in Figure A.3 (Appendix A.3).

Recovery Inconsistencies in Transition Probabilities

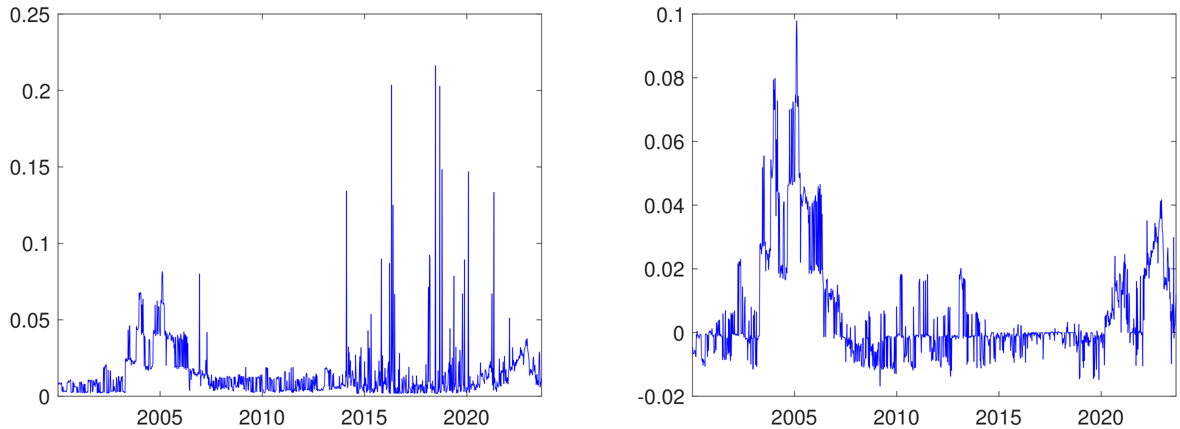
For each date in the sample, we record the recovered one-period transition probabilities $\{\overline{p}_{\bar{i}\bar{j}}\}$ between states $\bar{i}$ and $\bar{j}$ in the consolidated model and $\{p_{ij}\}$ between states i and j in the full model.[32] We compute the aggregate one-period transition probability $p_{i\bar{j}}$ from an original state i to a consolidated state $\bar{j}$ in the full model by summing over corresponding transitions. Below, we define the one-period relative probability differences associated with the transitions to a consolidated state $\bar{j}$ from the current state in the two models

$$dp_{\bar{1}\bar{j}} \equiv \frac{\overline{p}_{\bar{1}\bar{j}} - \sum_{j \in \bar{j}} p_{1j}}{\sum_{j \in \bar{j}} p_{1j}}, \qquad \bar{j} \in \overline{\mathcal{S}}. \tag{32}$$

The consistency condition (Definition 1) posits that these relative probability differences vanish for consistent recoveries of the full and consolidated models. The magnitude of $dp_{\bar{1}\bar{j}}$ therefore proxies for the inconsistencies in the recovered probabilities across the two models.

Panel A of Figure 2 plots the time series of the mean of the absolute values of relative probability difference, $|dp_{\bar{1}\bar{j}}|$, across all consolidated states $\bar{j} \in \overline{\mathcal{S}}$ for each date in our sample.

[32]Recall that the Recovery Theorem recovers the entire transition matrix $\mathbf{P}$ under the physical measure, including transitions from a non-current state i (5).



(A) Mean of absolute value of relative probability difference

(B) Median of relative probability difference

Figure 2: Relative probability difference between the consolidated and full models

Notes: This figure plots the mean and median of the relative differences of the one-month recovered transition probabilities between the (150-state) consolidated and (601-state) full models, starting from the current state to states in the consolidated model. Panel A plots the mean of the absolute values of the relative difference, and Panel B plots the median of the relative difference. The relative probability difference is defined as $\frac{\overline{p}_{\overline{1}\overline{j}} - \sum_{j \in \overline{j}} p_{1j}}{\sum_{j \in \overline{j}} p_{ij}}$, where $\overline{j} \in \overline{\mathcal{S}}$ and $j \in \mathcal{S}$, and $\overline{p}_{\overline{1}\overline{j}}$ and p_{1j} denote the recovered one-month transition probabilities from the current state of the consolidated and full models, respectively. The sample period is every Wednesday from January 5, 2000 to August 30, 2023.

This time series quantifies the unsigned average deviation from the consistent baseline of the recovered transition probabilities from the current state. Panel B plots the median of $dp_{\overline{1}\overline{j}}$ across $\overline{j} \in \overline{\mathcal{S}}$ for each date, which quantifies the signed average deviation of the recovered transition probabilities. Both panels show a non-vanishing relative probability difference for many dates of the sample period from January 2000 to August 2023, indicating persistent inconsistencies of the recovered transition probabilities. In terms of magnitude, the median value of the relative difference is large around 2005 and during the recent periods after 2020, while the mean value is larger between 2015 and 2020, indicating a skewed distribution of the inconsistencies in these periods. We further present two time-series examples of $dp_{\overline{1}\overline{j}}$ for the states $\overline{j}_{\min}$ and $\overline{j}_{\max}$ of the consolidated model's minimum and maximum recovered marginal utilities in Figure A.1 (Appendix A.3). The figure exhibits persistent inconsistencies in the recovered transition probabilities, with larger inconsistencies associated with the adverse state $\bar{j}_{\max}$, similar to the pattern observed in Figure 1.

Variations in Recovery Inconsistencies

In this section, we consider the variations in the inconsistencies of recovered marginal utilities and recovered transition probabilities.

Variable inconsistencies in recovered marginal utilities: To analyze the variations of recovery inconsistencies in marginal utilities with how well the condition for recovery consistency is met in the data, we introduce two empirical measures quantifying the recovery consistency associated with marginal utilities. The first consistency measure $\mathcal{C}_{\bar{j}}$ is a local measure, defined for a consolidated state $\bar{j} \in \overline{\mathcal{S}}$ as the difference between the maximum and minimum marginal utilities associated with original states j belonging to $\bar{j}$. Below, we consider two specific choices for the consolidated state $\bar{j}$, namely, $\bar{j}_{\min}$ and $\bar{j}_{\max}$. The second measure $\mathcal{C}_g$ is a global measure, defined as the standard deviation of marginal utilities across all $S = 601$ states in the full model,

$$\mathcal{C}_{\bar{j}} \equiv \max_{j \in \bar{j}} M_j - \min_{j \in \bar{j}} M_j, \qquad \bar{j} \in \{\bar{j}_{\max}, \bar{j}_{\min}\} \subset \overline{\mathcal{S}}, \qquad\qquad \mathcal{C}_g \equiv \sqrt{Var\left(M_j\right)}\Big|_{j \in \mathcal{S}}. \tag{33}$$

The condition for recovery consistency (Definition 1) posits identical M_j for all $j \in \bar{j}$, implying that $\mathcal{C}_{\bar{j}_{\max}} = 0$ and $\mathcal{C}_{\bar{j}_{\min}} = 0$.[33] As a result, the (non-vanishing) magnitudes of $\mathcal{C}_{\bar{j}_{\max}}$ and $\mathcal{C}_{\bar{j}_{\min}}$ indicate the presence and severity of inconsistencies in the recovered risk preferences locally at $\bar{j}_{\max}$ and $\bar{j}_{\min}$. Similarly, when the underlying market model features a significant state-dependent dynamics (i.e., highly variable $\{M_j\}$), the consolidation of $\mathcal{S}$ into a coarser $\overline{\mathcal{S}}$ needs to account for this variability to preserve the consistency (discussed below Equation (28)). As a result, a sizable magnitude of $\mathcal{C}_g$ exposes the simplicity of consolidation (grouping every four of the original 601 states in $\mathcal{S}$ into one of the 150 consolidated states in $\overline{\mathcal{S}}$) against data, entailing sizable recovery inconsistencies (globally across the state space).

[33] In fact, consistent recovery implies that $\mathcal{C}_{\bar{j}} = 0, \forall \bar{j} \in \overline{\mathcal{S}}$.

To examine these relationships, Panel A of Table 3 reports the point estimates and the statistical significance of the time-series correlations between the consistency measures (33) and the degrees of recovered marginal utility inconsistencies. The latter are quantified by the deviations of the minimum and maximum recovered marginal utilities (in the consolidated model) from the median recovered marginal utilities (in the full model) among those associated with corresponding original states, $|\overline{M}_{\overline{j}_{\min}} - M_{j_{\text{med}}}|$, $j_{\text{med}} \in \overline{j}_{\min}$, and $|\overline{M}_{\overline{j}_{\max}} - M_{j_{\text{med}}}|$, $j_{\text{med}} \in \overline{j}_{\max}$, for each date in our sample. The global measure $\mathcal{C}_g$ positively and statistically significantly (p-value < 0.01) correlates with both inconsistency proxies, and the local measure $\mathcal{C}_{\overline{j}}$ positively and statistically significantly (p-value < 0.01) correlates with the deviation concerning $\overline{M}_{\overline{j}_{\min}}$.[34] These empirical results provide supporting evidence to the relationships discussed earlier that larger inconsistencies in the recovered risk preferences tend to take place when the consistency condition is violated more significantly.

Variable inconsistencies in recovered transition probabilities: Similarly, we analyze the variations of recovery inconsistencies in transition probabilities with how well the recovery consistency condition (Definition 1) is met in the data. This consistency condition posits identical transition probabilities starting from any original states i, k belonging to a consolidated state $\overline{j}$ to another consolidated state $\overline{h}$, or, $p_{i\overline{h}} = p_{k\overline{h}}$, $\forall i, k \in \overline{j}$ and $\overline{j}, \overline{h} \in \overline{\mathcal{S}}$, for consistent recoveries. Accordingly, we employ the empirical deviation from these identities to construct a local consistency measure based on one-month and one-year transition probabilities from the current state to the consolidated state $\overline{j} \in \{\overline{j}_{\max}, \overline{j}_{\min}\}$ as follows

$$\mathcal{C}^P_{1;\overline{1}\,\overline{j}} \equiv \max_{h \in \overline{1}} p_{1;\,h\overline{j}} - \min_{h \in \overline{1}} p_{1;\,h\overline{j}}, \quad \mathcal{C}^P_{12;\overline{1}\,\overline{j}} \equiv \max_{h \in \overline{1}} p_{12;\,h\overline{j}} - \min_{h \in \overline{1}} p_{12;\,h\overline{j}}, \quad \overline{j} \in \{\overline{j}_{\max}, \overline{j}_{\min}\} \subset \overline{\mathcal{S}}, \tag{34}$$

where $\overline{1}$ denotes the current state in the consolidated model, $p_{1;\,h\overline{j}} = \sum_{j \in \overline{j}} p_{1;\,hj}$, $p_{12;\,h\overline{j}} = \sum_{j \in \overline{j}} p_{12;\,hj}$, and $\{p_{1;\,hj}\}$ and $\{p_{12;\,hj}\}$ are respectively the one-month and one-year recovered

[34] The local measure $\mathcal{C}_{\overline{j}_{\max}}$ correlates slightly negatively with the deviation concerning $\overline{M}_{\overline{j}_{\max}}$, but the point estimate is statistically insignificant.

Table 3: Variations in recovery inconsistencies

Panel A: Recovered marginal utilities		
	$\lvert \overline{M}_{\overline{j}_{\min}} - M_{j_{\mathrm{med}}} \rvert$	$\lvert \overline{M}_{\overline{j}_{\max}} - M_{j_{\mathrm{med}}} \rvert$
$\mathcal{C}_{\overline{j}_{\min}}$	0.27***	
	(9.75)	
$\mathcal{C}_{\overline{j}_{\max}}$		-0.01
		(-0.49)
$\mathcal{C}_g$	0.24***	0.20***
	(8.70)	(7.13)

Panel B: Recovered transition probabilities, one-month and one-year ahead				
	$\lvert \overline{p}_{1;\,\overline{1}\,\overline{j}_{\min}} - p_{1;\,1\overline{j}_{\min}} \rvert$	$\lvert \overline{p}_{1;\,\overline{1}\,\overline{j}_{\max}} - p_{1;\,1\overline{j}_{\max}} \rvert$	$\lvert \overline{p}_{12;\,\overline{1}\,\overline{j}_{\min}} - p_{12;\,1\overline{j}_{\min}} \rvert$	$\lvert \overline{p}_{12;\,\overline{1}\,\overline{j}_{\max}} - p_{12;\,1\overline{j}_{\max}} \rvert$
$\mathcal{C}^P_{1\,;\overline{1}\,\overline{j}_{\min}}$	0.48***			
	(18.97)			
$\mathcal{C}^P_{1\,;\overline{1}\,\overline{j}_{\max}}$		0.32***		
		(11.73)		
$\mathcal{C}^P_{12\,;\overline{1}\,\overline{j}_{\min}}$			0.33***	
			(12.33)	
$\mathcal{C}^P_{12\,;\overline{1}\,\overline{j}_{\max}}$				0.69***
				(33.73)

Notes: This table reports the time-series correlations between the consistency measures and degrees of recovery inconsistencies for every Wednesday from January 5, 2000 to August 30, 2023. The local consistency measure $\mathcal{C}_{\overline{j}}$ for marginal utilities is the difference between the maximum and minimum marginal utilities associated with original states j that belong to $\overline{j}_{\max}$ or $\overline{j}_{\min}$. The global inconsistency measure $\mathcal{C}_g$ is the standard deviation of marginal utilities across all $S = 601$ states in the full model. The local consistency measure $\mathcal{C}^P_{\tau\,;\overline{1}\,\overline{j}}$ for transition probabilities of one-month ($\tau = 1$) and one-year ($\tau = 12$) horizons is the difference between the maximum and minimum transition probabilities of corresponding horizons starting from an original state belonging to the current consolidated state $\overline{1}$ to $\overline{j}_{\max}$ or $\overline{j}_{\min}$. t-statistics are reported in parentheses. * indicates significance at the 10% level; **, at the 5% level; and ***, at the 1% level.

transition probabilities between original states $h, j \in \mathcal{S}$ of the full model.[35] The (non-vanishing) magnitude of measures (34) indicates the presence and severity of recovery inconsistencies for these time horizons (and at the target state $\overline{j} \in \{\overline{j}_{\min}, \overline{j}_{\max}\}$). Panel B of Table 3 reports the point estimates and the statistical significance of the time-series correlations between the consistency measures (34) and the degrees of recovered probability inconsistencies of commensurate horizons. The latter are quantified by the difference between the

[35]Note that $\{p_{1;\,hj}\}$ (and $\{p_{12;\,hj}\}$) is the (h, j) element of the recovered one-month (and one-year) transition probability matrix $\mathbf{P}$ (and $\mathbf{P}^{12}$) of the full model.

probabilities (in the consolidated and full models) of the transitions from the current state to the states $\bar{j}_{\min}$ and $\bar{j}_{\max}$, namely, $|\overline{p}_{1;\,\bar{1}\bar{j}_{\min}} - \sum_{j\in\bar{j}_{\min}} p_{1;\,1\,j}|$, and $|\overline{p}_{1;\,\bar{1}\bar{j}_{\max}} - \sum_{j\in\bar{j}_{\max}} p_{1;\,1\,j}|$ (and similar expressions for the difference of probabilities for one-year transitions). The point estimates of all correlations are positive and statistically significant (p-value < 0.01), indicating that larger inconsistencies in recovered transition probabilities tend to take place when the violation of consistency condition is more significant. These empirical results hold for both one-month and one-year horizons, lending supporting evidence to inconsistencies in the recovered probabilities and their model-implied relationships. Further empirical evidence for variations of recovery inconsistencies with measures of consistency condition's violation for the benchmark period is presented in Table A.3 (Appendix A.3).

5 Conclusion

This paper examines the consistency aspects of recovering market's belief and time and risk preferences from asset prices. We identify an inherent endogeneity issue in the recovery framework: the state space specification is not observed yet required prior to the recovery process. Different subjective specifications lead to mutually inconsistent recovery results unless a strong necessary and sufficient condition is satisfied. Intuitively, this consistency condition ensures that no information about the underlying market model is lost in the recovery implementations under different specifications. Moreover, a finer partition of a subjective specification does not guarantee improved consistency in recovery results, as a finer specification is not necessarily more consistent with the underlying (true) model. Analytically, we deliver these results through a tractable perturbative analysis. Empirically, we document robust recovery inconsistencies, the magnitude of which increases with measures quantifying violations of the consistency condition, using option price data across different specifications and sample periods. Our findings demonstrate that an unambiguous and consistent recovery of the underlying market's characteristics from asset prices remains challenging.

References

Aït-Sahalia, Y. and Lo, A. W. (1998), 'Nonparametric estimation of state-price densities implicit in financial asset prices', *The Journal of Finance* **53**(2), 499–547.

Alvarez, F. and Jermann, U. (2005), 'Using asset prices to measure the persistence of the marginal utility of wealth', *Econometrica* **73**(6), 1977–2016.

Audrino, F., Huitema, R. and Ludwig, M. (2021), 'An empirical implementation of the Ross recovery theorem as a prediction device', *Journal of Financial Econometrics* **19**, 291–312.

Bakshi, G., Chabi-Yo, F. and Gao, X. (2018), 'A recovery that we can trust? Deducing and testing the restrictions of the recovery theorem', *The Review of Financial Studies* **31**(2), 532–555.

Borovička, J., Hansen, L. P. and Scheinkman, J. A. (2016), 'Misspecified Recovery', *The Journal of Finance* **71**(6), 2493–2544.

Borovička, J. and Stachurski, J. (2020), 'Necessary and sufficient conditions for existence and uniqueness of recursive utilities', *Journal of Finance* **75**(3), 1457–1493.

Breeden, D. T. and Litzenberger, R. H. (1978), 'Prices of state-contingent claims implicit in option prices', *The Journal of Business* **51**(4), 621–651.

Carr, P. and Yu, J. (2012), 'Risk, return, and Ross recovery', *The Journal of Derivatives* **20**(1), 38–59.

Dillschneider, Y. and Maurer, R. (2019), 'Functional Ross recovery: Theoretical results and empirical tests', *Journal of Economic Dynamics and Control* **108**, 103750.

Foresee, F. D. and Hagan, M. T. (1997), Gauss-newton approximation to bayesian learning, *in* 'Proceedings of international conference on neural networks (ICNN'97)', Vol. 3, IEEE, pp. 1930–1935.

Giglio, S., Maggiori, M., Stroebel, J. and Utkus, S. (2022), 'Five Facts about Beliefs and Portfolios', *American Economic Review* **111**, 1481–1522.

Gormsen, N. J. and Koijen, R. S. J. (2020), 'Coronavirus: Impact on Stock Prices and Growth Expectations', *Review of Asset Pricing Studies* **10**, 574–597.

Hansen, L. P. and Scheinkman, J. A. (2017), *Stochastic Compounding and Uncertain Valuation*, University of Chicago Press, pp. 21–50.

Jackwerth, J. C. and Menner, M. (2020), 'Does the Ross recovery theorem work empirically?', *Journal of Financial Economics* **137**(3), 723–739.

Jensen, C. S., Lando, D. and Pedersen, L. H. (2019), 'Generalized Recovery', *Journal of Financial Economics* **133**(1), 154–174.

Ludwig, M. (2015), 'Robust estimation of shape-constrained state price density surfaces', *The Journal of Derivatives* **22**(3), 56–72.

Man, Y.-K. (2017), 'On computing the vandermonde matrix inverse', *Proceedings of the World Congress on Engineering, WCE 2017 London, U.K.* **1**.

Martin, I. (2017), 'What is the expected return on the market?', *The Quarterly Journal of Economics* **132**, 367–433.

Martin, I. W. and Ross, S. A. (2019), 'Notes on the yield curve', *Journal of Financial Economics* **134**(3), 689–702.

Nguyen, D. and Widrow, B. (1990), Improving the learning speed of 2-layer neural networks by choosing initial values of the adaptive weights, *in* '1990 IJCNN international joint conference on neural networks', IEEE, pp. 21–26.

Qin, L. and Linetsky, V. (2016), 'Positive eigenfunctions of Markovian pricing operators: Hansen-Scheinkman factorization, Ross recovery, and long-term pricing', *Operations Research* **64**(1), 99–117.

Qin, L., Linetsky, V. and Nie, Y. (2018), 'Long forward probabilities, recovery, and the term structure of bond risk premiums', *The Review of Financial Studies* **31**(12), 4863–4883.

Ross, S. (2015), 'The Recovery Theorem', *The Journal of Finance* **70**(2), 615–648.

Walden, J. (2017), 'Recovery with unbounded diffusion processes', *Review of Finance* **21**(4), 1403–1444.

Online Appendices

(not intended for publication)

These online appendices provide supporting materials for the paper. Appendix A concerns data, empirical methodologies, robustness and further empirical results. Appendix B concerns technical derivations and further theoretical analysis.

A Data, Empirical Methodology and Further Results

We present details concerning option and interest rate data and their processing in Appendix A.1, empirical methodologies in Appendix A.2, further empirical results and analyses in Appendix A.3.

A.1 Data

Our data source is OptionMetrics IvyDB US database, which contains historical data on US listed index, ETF, and equity options. It also contains other relevant information such as interest rates and index dividend yields. In particular, we focus on options on S&P 500 (SPX) because they are among the most liquid and actively traded options.

A.1.1 Option Price Data

The sample period in our main empirical analysis is from January 5, 2000 to August 30, 2023, and we obtain call and put option prices for each Wednesday. If price data is unavailable for a particular Wednesday, potentially due to a holiday, we substitute it with the price from the previous trading day. We define moneyness to be the ratio of strike to spot price and keep only OTM options. We exclude options with the average best bid and best offer price less than \$0.50 or with missing price data. We also limit the options in our sample to a fixed moneyness domain of $m \in [0.4, 1.6]$ and days-to-maturity domain of $\tau \in [20, 730]$.

A.1.2 Interest Rate and Dividend Yield Data

We obtain both interest rate data and S&P 500 dividend yield data from OptionMetrics.[1] However, neither the interest rate nor the dividend yield data offers a comprehensive coverage of all combinations of dates and maturities (tenors), which is essential to compute the implied volatilities (IV) for options in our sample and the Black-Scholes prices for each point on the estimated IV surface.

For each date, we linearly interpolate the interest rates and dividend yields between quoted tenors. Since the available tenors in the data may fall within the interval of $[20, 730]$, we linearly extrapolate the data so that we obtain a full coverage of date and tenor pairs. It may be possible that the extrapolated data turn out to be negative. In such a case, we impose a lower bound of zero. We locally smooth interest rates and dividend yields so that the term structures do not contain kinks.

A.1.3 Data Processing

In order to apply the put-call parity to translate traded OTM puts into the corresponding ITM calls (which may not be traded), we would need to compute the implied forward prices for the date and tenor pairs of OTM puts. We first apply the method proposed by Aït-Sahalia and Lo (1998) and then supplement the data using the processed dividend yield data from OptionMetrics.

In Aït-Sahalia and Lo (1998), the forward prices are backed out from the put-call parity using the close to ATM option pairs. To do so, for each combination of date and tenor in our sample, we find pairs of call and put options that have the same strike price and have moneyness between 0.99 and 1.01. We then compute the forward price F via the put-call parity

$$C + Ke^{-r\tau} = P + Fe^{-r\tau}. \tag{A.1}$$

There could be multiple implied forward prices for a given pair of date and tenor as we consider a range of moneyness, albeit a small range. In such a case, we take the average of the implied forward prices. The dividend yields are then derived from the spot-forward parity.

[1] In 2024, OptionMetrics started to provide expiration/maturity dates for dividend yield data.

However, due to data availability, this method does not guarantee that forward prices and dividend yields exist for all date and tenor pairs. Since dividend yield is crucial for deriving implied volatilities, we address this issue by incorporating the dividend yield data provided by OptionMetrics, as mentioned above. After this step, we are able to translate all OTM puts in our sample into ITM calls using the put-call parity.

To finalize the data for further analyses, such as estimating IV surface, we have to ensure that the options in our sample do not violate the no-arbitrage restrictions. In particular, call option prices C must satisfy the price bounds (29) as well as constraints on vertical and butterfly spreads (30). Finally, we back out the implied volatilities using Black-Scholes formula for each option in our sample.

A.2 Empirical Methodology

We describe the neural network approach to generate implied volatility surfaces in Appendix A.2.1 and the associated recovery implementation in Appendix A.2.2.

A.2.1 Neural Network

We follow Ludwig (2015)'s approach of employing neural networks to estimate a stable implied volatility surface from the option data. Prior to estimating the model parameters in (31), we need to ensure the stability of the resulting surface at the tenor boundaries and prevent calendar arbitrage. Therefore, for each date, we augment our sample as follows. To prevent calendar arbitrage, we use the average IV of ATM calls (those with moneyness between 0.99 and 1.01) to create artificial option data for each grid point of moneyness between 1.2 and 1.6 and tenors between 10 and 20 days. To stabilize the results at the tenor boundaries, we keep the first string of options with $\tau > 730$ when processing the raw data, and extract the first string of options with $\tau \geq 20$ and repeat it for each grid point of $m \in [0.9, 1.1]$ at the tenor of 10 days.

Next, we compute the implied total variance for each option on a given date in our augmented data and estimate (31) to obtain the parameters for the implied volatility surface. We obtain the initial parameter values $\boldsymbol{\theta}_0$ of the neural network by following Nguyen and Widrow (1990) and set the number of neurons (i.e., non-linear basis expansions) to be $N = \max\{10, 10 + \lfloor Z \rfloor\}$ with $Z \sim \mathcal{N}(0, 2)$. We map the option prices in our sample to implied volatilities using the Black-Scholes formula and follow Foresee and Hagan (1997) to find the set of model parameters $\boldsymbol{\theta} = \{\beta_0\} \cup \{\alpha_{0i}, \alpha_{1i}, \alpha_{2i}, \beta_i\}_{i=1}^N$ that minimizes the residual sum of squares

$$\min_{\boldsymbol{\theta}} \; RSS(\boldsymbol{\theta}) = \min_{\boldsymbol{\theta}} \; \sum \omega(m)(\widehat{\nu}(m, \tau; \boldsymbol{\theta}) - \nu(m, \tau))^2, \tag{A.2}$$

where $\widehat{\nu}(m, \tau; \boldsymbol{\theta})$ is the fitted implied total variance (31) given $\boldsymbol{\theta}$ and $\nu(m, \tau)$ is from the data. The weights $\omega(m)$ are given by $\phi(m|1, 0.2) + \phi(m|1, 0.1)$, where $\phi(\cdot|\mu, \sigma)$ is the normal density function with mean μ and standard deviation σ.

We plug $\boldsymbol{\theta}_0$ into the objective function $RSS(\boldsymbol{\theta})$. After initialization, solve (A.2) for the optimal parameters via Foresee and Hagan (1997) in the following steps:

1. Calculate the Jacobian matrix $\mathbf{J} = \frac{\partial RRS(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}$ and the Hessian matrix $\mathbf{H} = \mathbf{J}'\mathbf{J}$;
2. Calculate the residual vector $\mathbf{e} \equiv [\sqrt{\omega(m)_i}(\widehat{\nu}_i(m, \tau; \boldsymbol{\theta}) - \nu_i(m, \tau)))]$, where i denotes the ith observation;
3. Compute the gradient vector $\mathbf{g} = 2\mathbf{e}'\mathbf{J}$;
4. Update the parameters $\boldsymbol{\theta}_{new} = \boldsymbol{\theta} - (\mathbf{H} + \lambda \mathbf{I})^{-1}\mathbf{g}$, where $\mathbf{I}$ is the identity matrix and λ is the damping parameter;
5. Evaluate the objective function at $\boldsymbol{\theta}_{new}$;
6. If $RSS(\boldsymbol{\theta}_{new}) < RSS(\boldsymbol{\theta})$, update the damping parameter to $\lambda_{new} = \lambda / v_1$; otherwise, update it to $\lambda_{new} = \lambda v_2$, where $v_1, v_2 > 1$;
7. Repeat the previous steps until convergence or the maximum number of iterations is reached.

We train each network for 100 iterations and obtain an IV surface for $m \in [0.4, 1.6]$ and $\tau \in [20, 730]$, where the grid size of m is chosen to be 0.002, i.e., a total of 601 states. After obtaining a solution $\boldsymbol{\theta}^*$, we produce the associated implied total variance $\widehat{\nu}(m, \tau; \boldsymbol{\theta}^*)$ and back out the call option prices using the Black-Scholes formula for all grid points on the surface. Then, we check whether the conditions (29), (30), as well as the restriction on calendar spread

$$\widehat{\nu}(m, \tau_1; \boldsymbol{\theta}^*) > \widehat{\nu}(m, \tau_2; \boldsymbol{\theta}^*) \ \text{ if } \ \tau_1 > \tau_2. \tag{A.3}$$

are satisfied for each of these call prices. If so, we call $\boldsymbol{\theta}^*$ a valid solution to the problem (A.2). We choose the 5 solutions with lowest residual sum of squares (A.2) out of the 15 valid solutions and take the average as our estimated IV surface.

A.2.2 Recovery Implementations

We detail the procedure to implement separate recoveries for different subjective specifications. For each specification, we follow Audrino et al. (2021)'s benchmark approach in employing a regularization procedure (ridge regression) to solve for a stable one-period AD price matrix and its inverse. We relate (i.e., consolidate) observable option price data under different specifications by the law of one price and use them as inputs to separate recovery implementations and obtain recovery results for respective specifications. In the difference with the benchmark approach, we do not construct or relate recovery results across different specifications by interpolations.

We consider a model with $S = 601$ states of moneyness and $T = 711$ tenors (days), i.e., $m \in [0.4, 1.6]$ with tick size of 0.002 and $\tau \in [20, 730]$. We name this as the full model. As a baseline, we estimate the monthly AD price matrix $\mathbf{A}$ for each date. Recall that the state price transition (i.e., one-period AD price) matrices satisfy (6) under the assumptions in Ross (2015).

To estimate $\mathbf{A}$, we would solve the following S least squares problems

$$\min_{\mathbf{A}_{:\,i} \geq \mathbf{0}} \ (\mathbf{A}_{\tau+1;:\,i} - \mathbf{A}_\tau \mathbf{A}_{:\,i})'(\mathbf{A}_{\tau+1;:\,i} - \mathbf{A}_\tau \mathbf{A}_{:\,i}), \ i = 1, \ldots, S, \tag{A.4}$$

where $\mathbf{A}_{:\,i}$ and $\mathbf{A}_{\tau+1;:\,i}$ denote the i-th column of $\mathbf{A}$ and $\mathbf{A}_{\tau+1}$, respectively. However, $\mathbf{A}_\tau$ is often ill-conditioned, which leads to unstable solutions even with small perturbations in $\mathbf{A}_\tau$. Hence, we include L^2 regularization via the ridge regression

$$\min_{\mathbf{A}_{:\,i} \geq \mathbf{0}} \ (\mathbf{A}_{\tau+1;:\,i} - \mathbf{A}_\tau \mathbf{A}_{:\,i})'(\mathbf{A}_{\tau+1;:\,i} - \mathbf{A}_\tau \mathbf{A}_{:\,i}) + \zeta \mathbf{A}'_{:\,i} \mathbf{A}_{:\,i}, \ i = 1, \ldots, S. \tag{A.5}$$

The key to solving (A.5) is to find the ridge parameter ζ. To do so, we choose $\zeta > 0$ to minimize the generalized Kullback-Leibler divergence $D_{KL}(\mathbf{A}_\tau \| \widehat{\mathbf{A}}_\tau)$ between the state prices $\mathbf{A}_\tau$ implied from the data and $\widehat{\mathbf{A}}_\tau$ generated by the Markov transition model (6),

$$D_{KL}(\mathbf{A}_\tau \| \widehat{\mathbf{A}}_\tau) = \sum_{j,\tau} \left[\mathbf{A}_{\tau;ij} \log\left(\frac{\mathbf{A}_{\tau;ij}}{\widehat{\mathbf{A}}_{\tau;ij}}\right) - (\mathbf{A}_{\tau;ij} - \widehat{\mathbf{A}}_{\tau;ij}) \right], \tag{A.6}$$

where $\mathbf{A}_{\tau;ij}$ and $\widehat{\mathbf{A}}_{\tau;ij}$ represent state prices paying off in state j with tenor τ, from the perspective of the current state i.

Specifically, we consider a grid of $\zeta \in (0, 0.3]$ and for each ζ on the grid, we solve (A.5) and obtain a transition matrix $\mathbf{A}(\zeta)$, which depends on ζ. We then compute the corresponding generalized Kullback-Leibler divergence $D_{KL}(\mathbf{A}_\tau \| \widehat{\mathbf{A}}_\tau; \zeta)$ defined in (A.6), which is minimized at the optimal ridge parameter ζ^*.[2] The AD transition matrix is therefore $\mathbf{A}(\zeta^*)$, which we then use to identify its dominant eigenspace $(\delta^{(1)}, \mathbf{x}^{(1,R)})$ and recover the physical transition probabilities and marginal utilities as follows

$$\delta = \delta^{(1)}, \qquad M_i = \frac{1}{x_i^{(1,R)}}, \qquad \mathbf{P} = \delta^{-1}\, \mathbf{Diag}(\mathbf{x}^{(1,R)})^{-1}\, \mathbf{A}(\zeta^*)\, \mathbf{Diag}(\mathbf{x}^{(1,R)}). \tag{A.7}$$

For the recovery of the consolidated model, we first sum up individual state prices within a consolidated state for a given tenor, using the law of one price. As an example, we choose the number of states to be 150 for the consolidated model, and sum up every 4 states in the full model. For each consolidated state $\bar{j}$, we set the moneyness $\overline{m}_{\bar{j}}$ to be the median moneyness of all individual states $j \in \bar{j}$. Then, we solve for the optimal ridge parameter $\bar{\zeta}^*$ and the corresponding 150×150 transition matrix $\overline{\mathbf{A}}(\bar{\zeta}^*)$ and subsequently recover the 150-state transition probability matrix $\overline{\mathbf{P}}$ and marginal utilities $\overline{\mathbf{M}}$.

As it may require considerable computing power to solve the non-negative ridge regression (A.5) for a large number of states, Audrino et al. (2021) propose an alternative method of recovering 1201-state probabilities and marginal utilities. Specifically, instead of estimating a full 1201×1201 matrix $\mathbf{A}$, they reduce the number of states to 150 and estimate a 150×150 matrix $\widetilde{\mathbf{A}}$ via (A.5) and the

[2] Following Audrino et al. (2021), we add 10^{-20} to $\mathbf{A}$ and $\widehat{\mathbf{A}}$ in (A.6) to avoid dividing by zero.

associated 150×1 dominant eigenvector $\widetilde{\mathbf{x}}^{(1,R)}$ is obtained. After finding out the optimal ridge parameter $\widetilde{\zeta}^*$, they obtain the 150-state dominant right eigenvector $\widetilde{\mathbf{x}}^{(1,R)}$ of the matrix $\widetilde{\mathbf{A}}(\widetilde{\zeta}^*)$. Subsequently, a cubic spline interpolation to $\widetilde{\mathbf{x}}^{(1,R)}$ is used to restore to 1201 states. Finally, the recovered 1201-state probabilities $\widetilde{\mathbf{P}}$ and marginal utilities $\widetilde{\mathbf{M}}$ are obtained via (A.7).

A.3 Further Empirical Results

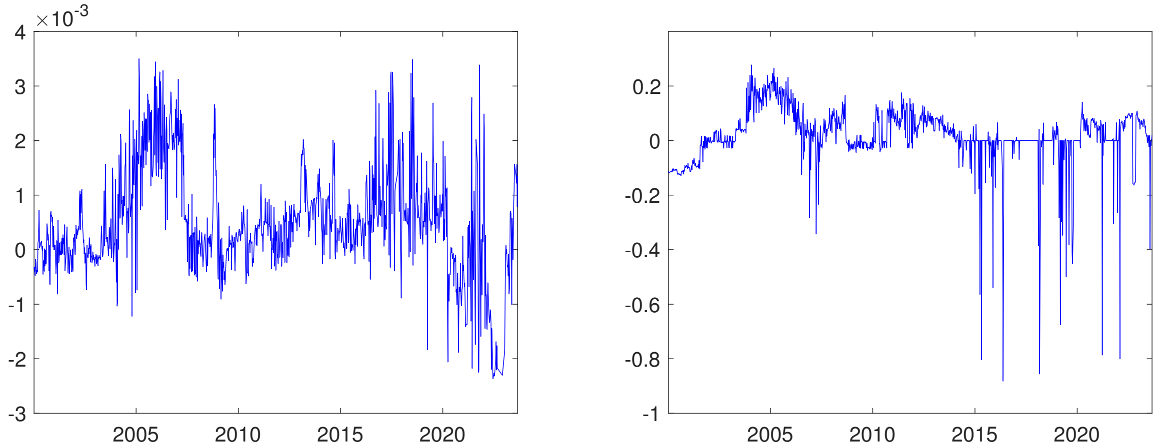
This appendix presents the robustness, replication and validation of methodology, and further empirical results of recovery inconsistencies concerning 601-state versus 150-state specifications (Appendix A.3.1), and the recovery implementation under the 1201-state (benchmark) specification (Appendix A.3.2).

A.3.1 Concerning 601-state and 150-state Specifications

In this appendix, we consider the 601-state (full model) versus 150-state (consolidated model) specifications as in the main text.

Robustness check (using median marginal utility): In Figure A.2 repeats the empirical exercise underlying Figure 1, but instead focuses on the median marginal utilities of the original states of the full model that correspond to either $\overline{M}_{\bar{\jmath}_{\min}}$ (Panel A) or $\overline{M}_{\bar{\jmath}_{\max}}$ (Panel B) of the consolidated model. Similar to Figure 1, we observe that there is a persistent difference between the marginal utilities of the two models, indicating inconsistencies in the recovered marginal utilities that are significant for the adverse state $\bar{\jmath}_{\max}$ (of maximum recovered marginal utility in the consolidated model, Panel B) and for the recent period (from 2014 onward).

Two time-series examples of relative probability differences (32): Panel A of Figure A.1 plots the time series of the relative probability difference $dp_{\bar{1}\bar{\jmath}_{\min}}$ associated with the state $\bar{\jmath}_{\min}$ (of minimum recovered marginal utility), and Panel B plots $dp_{\bar{1}\bar{\jmath}_{\max}}$ associated with $\bar{\jmath}_{\max}$ (of maximum recovered marginal utility). Both panels show a non-vanishing relative probability difference for many dates of the sample period from January 2000 to August 2023, indicating persistent inconsistencies of the recovered transition probabilities. In terms of magnitude, the relative probability difference



(A) Relative probability difference for minimum marginal utility state

(B) Relative probability difference for maximum marginal utility state

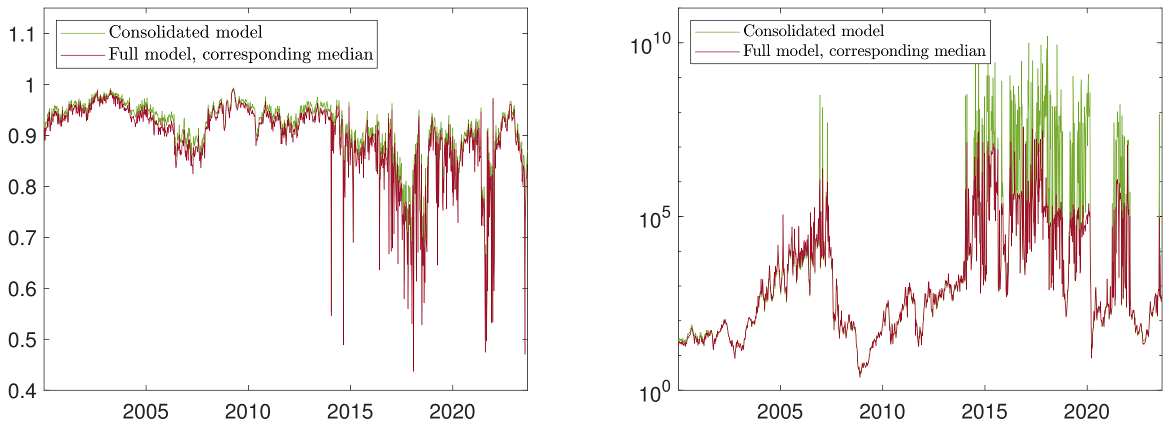
Figure A.1: Relative probability difference between the consolidated and full models

Notes: This figure plots the time series of relative differences of the one-month recovered transition probabilities between the (150-state) consolidated and (601-state) full models, starting from the current state to the minimum (Panel A) and the maximum (Panel B) marginal utility states in the consolidated model. The relative probability difference is defined as $\frac{\overline{p}_{\bar{1}\bar{j}} - \sum_{j \in \bar{j}} p_{1j}}{\sum_{j \in \bar{j}} p_{ij}}$, where $\bar{j} \in \overline{\mathcal{S}}$ and $j \in \mathcal{S}$, and $\overline{p}_{\bar{1}\bar{j}}$ and p_{1j} denote the recovered one-month transition probabilities from the current state of the consolidated and full models, respectively. The sample period is every Wednesday from January 5, 2000 to August 30, 2023.

associated with the transition to the adverse state $\bar{j}_{\max}$ in Panel B is significantly larger than that to the good state $\bar{j}_{\min}$ in Panel A. This pattern is also observed earlier in Figure 1 for the inconsistencies in recovered marginal utilities. Intuitively, it is consistent with the fact that adverse states of the underlying market tend to be rarer, they are more elusive to the recovery process. Further empirical evidence for inconsistencies in the recovered probabilities for the benchmark period is presented in Figures A.4 and A.5.

Validation of methodology (January 5, 2000 – December 26, 2012): Now, we limit our sample period to the benchmark sample period from January 5, 2000 to December 26, 2012, which consists of 197,771 call options, to validate our recovery implementations with the benchmark literature. Table A.1 shows the summary statistics of implied volatilities for various moneyness and tenors. The summary statistics are in line with those reported in Ludwig (2015). For completeness, we also report the risk-neutral and recovered moments of cross-sectional returns in Table A.2.[3]

[3] The results in this table are broadly comparable with the corresponding moments reported in Audrino et al. (2021), who employ 1201-state specification in place of our 601-state specification in Table A.2.



(A) Minimum marginal utilities of the consolidated model

(B) Maximum marginal utilities of the consolidated model

Figure A.2: Comparison of marginal utilities of the consolidated and full models

Notes: This figure plots the time series of the recovered minimum and maximum marginal utilities of the (150-state) consolidated model, along with the median marginal utilities among the original states of the (601-state) full model that correspond to the minimum and maximum marginal utilities states respectively in the consolidated model. Current state's marginal utility is normalized to one. The sample period is every Wednesday from January 5, 2000 to August 30, 2023.

Recovery inconsistencies (January 5, 2000 – December 26, 2012): We examine the recovery consistencies between the 601-state (full model) and 150-state (consolidated model) specifications for the benchmark sample period (January 5, 2000 – December 26, 2012). To this end, Figure A.3's Panel A (resp., Panel B) plots the time series of minimum (resp., maximum) recovered marginal utilities of the consolidated model at consolidated states $\overline{j}_{\min}$ (resp., $\overline{j}_{\max}$) against the corresponding minimum and maximum recovered marginal utilities of the full model among original states belonging to $\overline{j}_{\min}$ (resp., $\overline{j}_{\max}$). Figure A.4 further plots the mean and median of the relative probability difference (32), and Panels A and B of Figure A.5 provide two time-series examples of probabilities going to the minimum and maximum marginal utilities states of the consolidated model. Altogether, these plots (Figures A.3, A.4 and A.5) show persistent inconsistencies in both recovered marginal utilities and recovered transition probabilities. These patterns are similar to Figures 1 and 2 in the main text (which employ data of the entire sample period from January 5, 2000 – August 30, 2023), indicating the robustness of recovery inconsistencies.

Variations of recovery inconsistencies (January 5, 2000 – December 26, 2012): We examine the

Table A.1: Summary statistics of call options

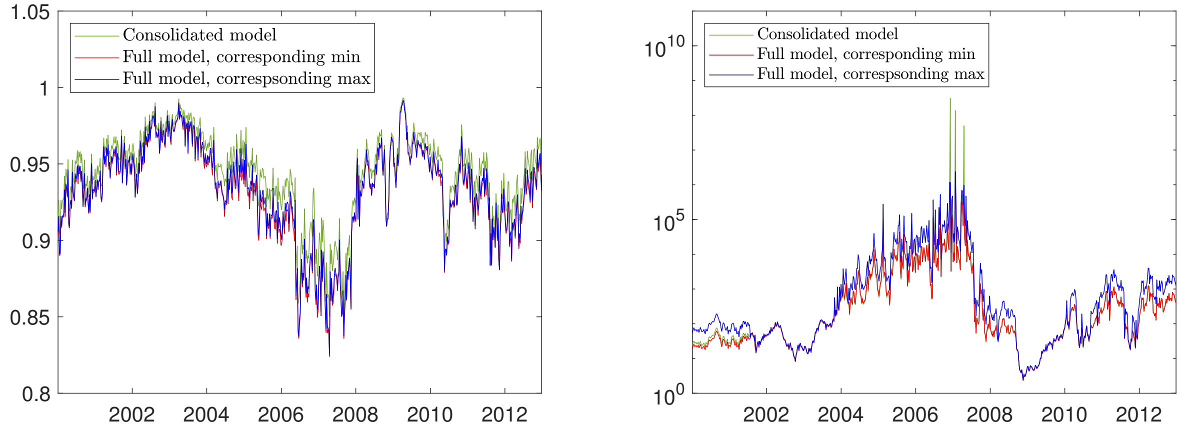
m	DITM < 0.90	ITM $[0.90, 0.99]$	ATM $(0.99, 1.01)$	OTM $[1.02, 1.10]$	DOTM > 1.10
Panel A: Maturity < 180 days					
IV (%)	34.92	22.28	18.72	17.33	21.06
Price	273.79	86.68	39.08	16.66	4.49
N	35,136	23,999	6,423	24,758	13,845
Panel B: Maturity 180 — 365 days					
IV (%)	30.94	22.27	20.67	19.16	18.55
Price	353.17	130.38	86.65	54.80	13.57
N	22,563	7,975	1,884	8,007	15,464
Panel C: Maturity > 365 days					
IV (%)	28.60	21.78	20.57	19.80	18.14
Price	390.73	164.85	123.99	92.80	28.88
N	15,277	4,939	1,158	4,723	11,620

Notes: This table reports the average implied volatilities (IV) and prices of the call options on S&P 500 in our sample for different categories of moneyness m and maturities. Our sample contains option data for every Wednesday from January 5, 2000 to December 26, 2012. Moneyness is defined as the ratio of strike to spot price.

Table A.2: Summary statistics of risk-neutral and recovered moments

	Median	Std dev	Min	25th	75th	Max
Panel A: Risk-neutral moments						
Mean (%)	−1.70	4.70	−29.69	−4.31	1.80	5.07
Volatility (%)	21.32	8.87	10.44	16.85	26.54	73.88
Skewness	−1.22	0.40	−2.71	−1.55	−0.97	−0.47
Kurtosis	7.24	3.32	3.85	5.63	9.61	23.60
Panel B: Recovered moments						
Mean (%)	9.80	4.63	−3.82	7.24	13.38	26.87
Volatility (%)	17.60	7.60	9.04	13.63	21.74	66.99
Skewness	−0.90	0.24	−1.89	−1.09	−0.75	−0.37
Kurtosis	5.41	1.46	3.74	4.72	6.58	12.46

Notes: This table reports the summary statistics of the risk-neutral and recovered moments of 30-day-to-maturity cross-sectional returns on S&P 500 for the (601-state) full model. Returns are computed by considering all possible values of S&P 500 on the grid in the next period. Mean and volatility are annualized and reported as percentages. All moments are unconditional over the period of January 5, 2000 to December 26, 2012.



(A) Minimum marginal utilities of the consolidated model

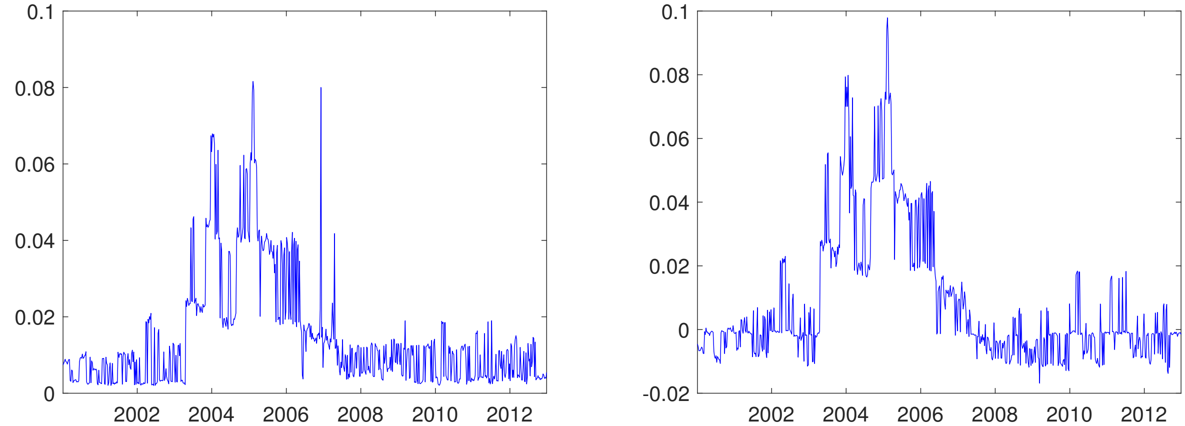
(B) Maximum marginal utilities of the consolidated model

Figure A.3: Comparison of marginal utilities of the consolidated and full models

Notes: This figure plots the time series of the recovered minimum ($\overline{M}_{\overline{j}_{\min}}$) and maximum ($\overline{M}_{\overline{j}_{\max}}$) marginal utilities of the (150-state) consolidated model, along with the minimum ($M_{j_{\min}}$) and maximum ($M_{j_{\min}}$) marginal utilities among the original states of the (601-state) full model that correspond to the minimum ($j_{\min} \in \overline{j}_{\min}$) and maximum ($j_{\max} \in \overline{j}_{\max}$) marginal utilities states respectively in the consolidated model. Current state's marginal utility is normalized to one. The sample period is every Wednesday from January 5, 2000 to December 26, 2012.

variation of recovery inconsistencies with measures quantifying violation of the recovery consistency condition for the benchmark sample period. Panel A (resp., Panel B) of Table A.3 reports the point estimates and the statistical significance of the time-series correlations between the local and global consistency measures $\mathcal{C}_{\overline{j}}$, $\mathcal{C}_g$ (33) (resp., the one-month and one-year consistency measures $\mathcal{C}^P_{1\,;\overline{1}\,\overline{j}}$, $\mathcal{C}^P_{12\,;\overline{1}\,\overline{j}}$ (34)) and the inconsistency level of recovered marginal utilities (resp., the inconsistency level of recovered transition probabilities of commensurate horizons). All but one of these correlation estimates are positive and statistically significant (p-value < 0.01), indicating that larger inconsistencies of recovered marginal utilities and transition probabilities tend to take place when the violation of consistency condition is more significant.[4] These findings for the benchmark sample period are similar to those for the entire sample period reported in Table 3 in the main text, signifying the robustness of these theory-implied variations of recovery inconsistencies in different sample periods.

[4]The only statistically insignificant point estimate is positive.



(A) Mean of absolute value of relative probability difference

(B) Median of relative probability difference

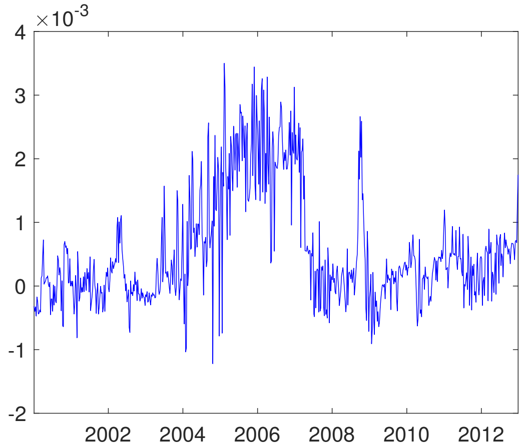
Figure A.4: Relative probability difference between the consolidated and full models

Notes: This figure plots the mean and median of the relative differences of the one-month recovered transition probabilities between the (150-state) consolidated and (601-state) full models, starting from the current state to states in the consolidated model. Panel A plots the mean of the absolute values of the relative difference, and Panel B plots the median of the relative difference. The relative probability difference is defined as $\frac{\overline{p}_{\overline{1}\overline{j}} - \sum_{j \in \overline{j}} p_{1j}}{\sum_{j \in \overline{j}} p_{ij}}$, where $\overline{j} \in \overline{\mathcal{S}}$ and $j \in \mathcal{S}$, and $\overline{p}_{\overline{1}\overline{j}}$ and p_{1j} denote the recovered one-month transition probabilities from the current state of the consolidated and full models, respectively. The sample period is every Wednesday from January 5, 2000 to December 26, 2012.

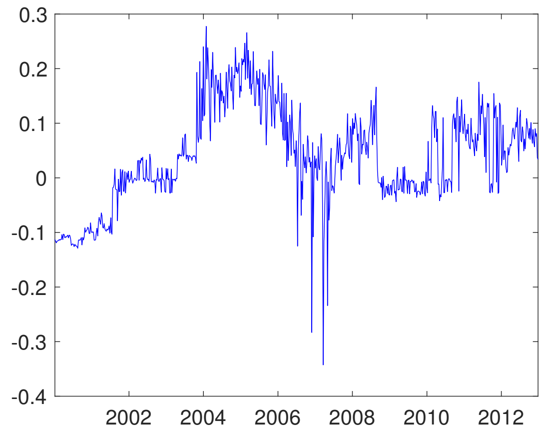
A.3.2 Concerning 1201-state and 150-state Specifications

In this appendix, for completeness and robustness, we consider the 1201-state (full model) versus 150-state (consolidated model) specifications of the benchmark literature, and for the entire sample period January 5, 2000 to August 30, 2023. Unlike the literature, we examine and demonstrate recovery inconsistencies by implementing two separate recoveries for these two models, while relating (consolidating) observable price inputs to these models by the law of one price (Section A.2.2). Specifically, for the 1201-state model, we solve the entire 1201 ridge regressions (A.5) (without lowering the number of states to a coarser model first and interpolating back to 1201-state model) to obtain the 1201×1201 AD price matrix $\mathbf{A}$. We then recover the time preference as well as the 1201-state marginal utilities and transition probabilities via (A.7). Table A.4 reports the risk-neutral and recovered moments of cross-sectional returns.

For the 150-state model, we employ the consolidated observable price data and solve (new) 150 ridge regressions (A.5) to obtain a 150×150 AD price matrix $\mathbf{A}$ and its dominant eigenvalue and



(A) Relative probability difference for minimum marginal utility state



(B) Relative probability difference for maximum marginal utility state

Figure A.5: Relative probability difference between the consolidated and full models

Notes: This figure plots the time series of relative differences of the one-month recovered transition probabilities between the (150-state) consolidated and (601-state) full models, starting from the current state to the minimum (Panel A) and the maximum (Panel B) marginal utility states in the consolidated model. The relative probability difference is defined as $\frac{\overline{p}_{\overline{1}\,\overline{j}}-\sum_{j\in\overline{j}}p_{1j}}{\sum_{j\in\overline{j}}p_{ij}}$, where $\overline{j}\in\overline{\mathcal{S}}$ and $j\in\mathcal{S}$, and $\overline{p}_{\overline{1}\,\overline{j}}$ and p_{1j} denote the recovered one-month transition probabilities from the current state of the consolidated and full models, respectively. The sample period is every Wednesday from January 5, 2000 to December 26, 2012.

eigenvector (which characterize the recovered time and risk preferences in the consolidated model).

Figure A.6 plots the time series of minimum and maximum marginal utilities of the consolidated model and the corresponding minimum and maximum marginal utilities of the full model within the states $\overline{j}_{\min}$ and $\overline{j}_{\max}$. We observe that the difference between $M_{j_{\min}}$ and $M_{j_{\max}}$ continues to persist over time. Moreover, $\overline{M}_{\overline{j}_{\min}}$ and $\overline{M}_{\overline{j}_{\max}}$ are outside the range of the corresponding $M_{j_{\min}}$ and $M_{j_{\max}}$ for most of the dates.

Figure A.7 further plots the mean and median of relative probability differences (32) concerning 1201-state and 150-state models. These plots exhibit non-vanishing values persisting over time. These patterns of marginal utilities and probabilities are similar to Figures 1 and 2 (concerning 601-state and 150-state models), indicating robust inconsistencies in recovered marginal utilities and transition probabilities across various specifications.

To examine the variation of the recovery consistencies with measures quantifying the violation of the recovery consistency condition concerning 1201-state and 150-state models, we estimate the

Table A.3: Variations in recovery inconsistencies

Panel A: Recovered marginal utilities		
	$\lvert\overline{M}_{\bar{j}_{\min}} - M_{j_{\text{med}}}\rvert$	$\lvert\overline{M}_{\bar{j}_{\max}} - M_{j_{\text{med}}}\rvert$
$\mathcal{C}_{\bar{j}_{\min}}$	0.67***	
	(23.59)	
$\mathcal{C}_{\bar{j}_{\max}}$		0.01
		(0.23)
$\mathcal{C}_g$	0.27***	0.85***
	(7.16)	(41.46)

Panel B: Recovered transition probabilities, one-month and one-year ahead				
	$\lvert\overline{p}_{1;\bar{1}\bar{j}_{\min}} - p_{1;1\bar{j}_{\min}}\rvert$	$\lvert\overline{p}_{1;\bar{1}\bar{j}_{\max}} - p_{1;1\bar{j}_{\max}}\rvert$	$\lvert\overline{p}_{12;\bar{1}\bar{j}_{\min}} - p_{12;1\bar{j}_{\min}}\rvert$	$\lvert\overline{p}_{12;\bar{1}\bar{j}_{\max}} - p_{12;1\bar{j}_{\max}}\rvert$
$\mathcal{C}^P_{1;\bar{1}\bar{j}_{\min}}$	0.54***			
	(16.57)			
$\mathcal{C}^P_{1;\bar{1}\bar{j}_{\max}}$		0.99***		
		(179.19)		
$\mathcal{C}^P_{12;\bar{1}\bar{j}_{\min}}$			0.29***	
			(7.73)	
$\mathcal{C}^P_{12;\bar{1}\bar{j}_{\max}}$				0.69***
				(24.58)

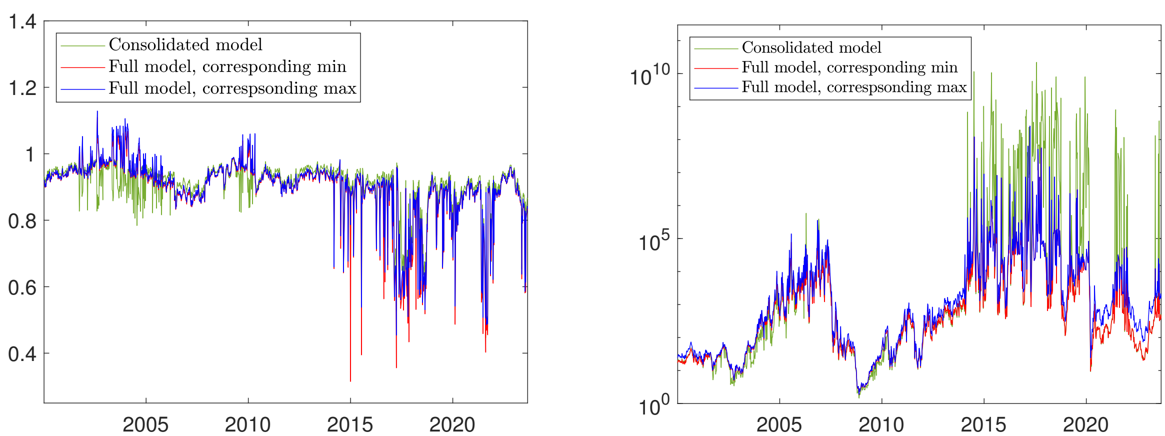
Notes: This table reports the time-series correlations between the consistency measures and degrees of recovery inconsistencies for every Wednesday from January 5, 2000 to December 26, 2012. The local consistency measure $\mathcal{C}_{\bar{j}}$ for marginal utilities is the difference between the maximum and minimum marginal utilities associated with original states j that belong to $\bar{j}_{\max}$ or $\bar{j}_{\min}$. The global inconsistency measure $\mathcal{C}_g$ is the standard deviation of marginal utilities across all $S = 601$ states in the full model. The local consistency measure $\mathcal{C}^P_{\tau;\bar{1}\bar{j}}$ for transition probabilities of one-month ($\tau = 1$) and one-year ($\tau = 12$) horizons is the difference between the maximum and minimum transition probabilities of corresponding horizons starting from an original state belonging to the current consolidated state $\bar{1}$ to $\bar{j}_{\max}$ or $\bar{j}_{\min}$. t-statistics are reported in parentheses. * indicates significance at the 10% level; **, at the 5% level; and ***, at the 1% level.

time-series correlations between consistency measures $\mathcal{C}_{\bar{j}}$, $\mathcal{C}_g$ (33) and recovered marginal utilities' inconsistency level, and between measures $\mathcal{C}^P_{1;\bar{1}\bar{j}}$, $\mathcal{C}^P_{12;\bar{1}\bar{j}}$ (34) and recovered transition probabilities' inconsistency level. Table A.5 presents the point estimates and the statistical significance of these correlations using data of the entire sample period January 5, 2000 to August 30, 2023. Most of these correlation estimates are positive and statistically significant (p-value < 0.01), indicating that larger recovery inconsistencies tend to take place when the violation of consistency condition is more significant. These findings are similar to those reported in Table 3 in the main text (concerning 601-state and 150-state specifications), signifying the robustness of these theory-implied variations

Table A.4: Summary statistics of risk-neutral and recovered moments

	Median	Std dev	Min	25th	75th	Max
Panel A: Risk-neutral moments						
Mean (%)	−2.11	4.10	−43.05	−3.81	−0.07	5.03
Volatility (%)	19.76	8.29	9.26	15.73	24.72	80.05
Skewness	−1.49	0.72	−6.55	−2.06	−1.09	−0.45
Kurtosis	9.10	7.69	3.83	6.34	14.19	14.19
Panel B: Recovered moments						
Mean (%)	11.15	5.12	−4.09	7.82	14.74	37.25
Volatility (%)	15.39	7.24	4.99	12.20	20.16	68.01
Skewness	−1.09	0.50	−6.08	−1.39	−0.85	−0.36
Kurtosis	6.70	4.17	3.69	5.30	8.88	81.90

Notes: This table reports the summary statistics of the risk-neutral and recovered moments of 30-day-to-maturity cross-sectional returns on S&P 500 for the 1201-state model. Returns are computed by considering all possible values of S&P 500 on the grid in the next period. Mean and volatility are annualized and reported as percentages. All moments are unconditional over the period of January 5, 2000 to August 30, 2023.



(A) Minimum marginal utilities of the consolidated model

(B) Maximum marginal utilities of the consolidated model

Figure A.6: Comparison of marginal utilities of the consolidated and full models

Notes: This figure plots the time series of the recovered minimum ($\overline{M}_{\bar{j}_{\min}}$) and maximum ($\overline{M}_{\bar{j}_{\max}}$) marginal utilities of the (150-state) consolidated model, along with the minimum ($M_{j_{\min}}$) and maximum ($M_{j_{\min}}$) marginal utilities among the original states of the (1201-state) model that correspond to the minimum ($j_{\min} \in \bar{j}_{\min}$) and maximum ($j_{\max} \in \bar{j}_{\max}$) marginal utilities states respectively in the consolidated model. Current state's marginal utility is normalized to one. The sample period is every Wednesday from January 5, 2000 to August 30, 2023.

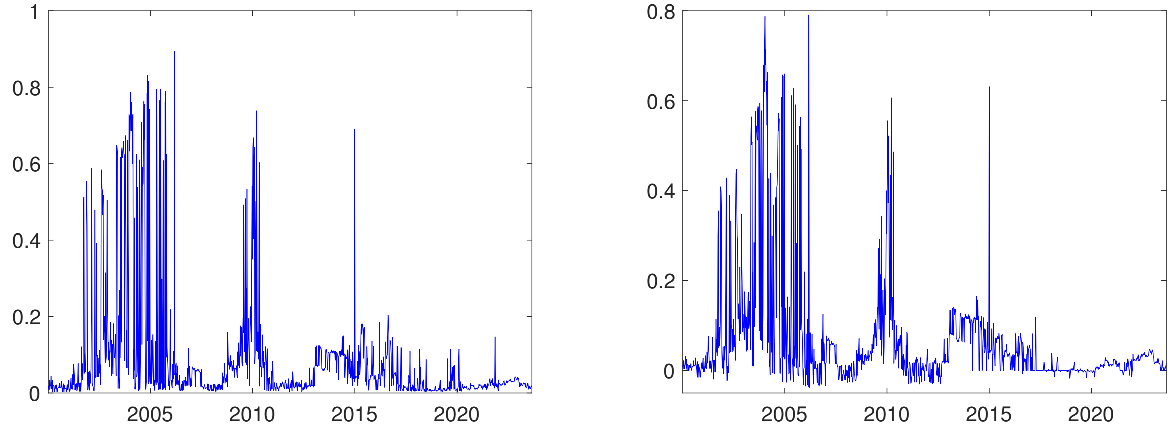
of recovery inconsistencies across various specifications.

Table A.5: Variations in recovery inconsistencies

Panel A: Recovered marginal utilities		
	$\lvert\overline{M}_{\bar{j}_{\min}} - M_{j_{\text{med}}}\rvert$	$\lvert\overline{M}_{\bar{j}_{\max}} - M_{j_{\text{med}}}\rvert$
$\mathcal{C}_{\bar{j}_{\min}}$	0.31***	
	(11.32)	
$\mathcal{C}_{\bar{j}_{\max}}$		-0.01
		(-0.25)
$\mathcal{C}_g$	0.11***	0.01
	(3.72)	(0.26)

Panel B: Recovered transition probabilities, one-month and one-year ahead				
	$\lvert\overline{p}_{1;\bar{1}\bar{j}_{\min}} - p_{1;1\bar{j}_{\min}}\rvert$	$\lvert\overline{p}_{1;\bar{1}\bar{j}_{\max}} - p_{1;1\bar{j}_{\max}}\rvert$	$\lvert\overline{p}_{12;\bar{1}\bar{j}_{\min}} - p_{12;1\bar{j}_{\min}}\rvert$	$\lvert\overline{p}_{12;\bar{1}\bar{j}_{\max}} - p_{12;1\bar{j}_{\max}}\rvert$
$\mathcal{C}^P_{1;\bar{1}\bar{j}_{\min}}$	0.61***			
	(26.84)			
$\mathcal{C}^P_{1;\bar{1}\bar{j}_{\max}}$		0.41***		
		(15.91)		
$\mathcal{C}^P_{12;\bar{1}\bar{j}_{\min}}$			0.94***	
			(96.73)	
$\mathcal{C}^P_{12;\bar{1}\bar{j}_{\max}}$				0.37***
				(13.93)

Notes: This table reports the time -series correlations between the consistency measures and degrees of recovery inconsistencies for every Wednesday from January 5, 2000 to August 30, 2023. The local consistency measure $\mathcal{C}_{\bar{j}}$ for marginal utilities is the difference between the maximum and minimum marginal utilities associated with original states j that belong to $\bar{j}_{\max}$ or $\bar{j}_{\min}$. The global inconsistency measure $\mathcal{C}_g$ is the standard deviation of marginal utilities across all $S = 1201$ states in the model. The local consistency measure $\mathcal{C}^P_{\tau;\bar{1}\bar{j}}$ for transition probabilities of one-month ($\tau = 1$) and one-year ($\tau = 12$) horizons is the difference between the maximum and minimum transition probabilities of corresponding horizons starting from an original state belonging to the current consolidated state $\bar{1}$ to $\bar{j}_{\max}$ or $\bar{j}_{\min}$. t-statistics are reported in parentheses. * indicates significance at the 10% level; **, at the 5% level; and ***, at the 1% level.



(A) Mean of absolute value of relative probability difference

(B) Median of relative probability difference

Figure A.7: Relative probability difference between the consolidated and full models

Notes: This figure plots the mean and median of the relative differences of the one-month recovered transition probabilities between the (150-state) consolidated and (1201-state) full models, starting from the current state to states in the consolidated model. Panel A plots the mean of the absolute values of the relative difference, and Panel B plots the median of the relative difference. The relative probability difference is defined as $\frac{\overline{p}_{\overline{1}\overline{j}} - \sum_{j \in \overline{j}} p_{1j}}{\sum_{j \in \overline{j}} p_{ij}}$, where $\overline{j} \in \overline{\mathcal{S}}$ and $j \in \mathcal{S}$, and $\overline{p}_{\overline{1}\overline{j}}$ and p_{1j} denote the recovered one-month transition probabilities from the current state of the consolidated and full models, respectively. The sample period is every Wednesday from January 5, 2000 to August 30, 2023.

B Technical Derivations and Analysis

We present a derivation of the necessary and sufficient condition for the recovery consistency in Appendix B.1, the least squares approach to recovery in Appendix B.2, and further quantitative analysis of recovery inconsistencies using Vandermonde matrix and their relationship with the spectral gap of AD price matrix in Appendix B.3.

B.1 Proof of Proposition 1

This appendix presents a proof of Proposition 1. The proof addresses separately whether the current state is a single or a coupled state.

Case 1 - Single Current State: We consider an original specification of $\mathcal{S} = \{1, \cdots, S\}$ and a consolidated specification $\overline{\mathcal{S}} = \{\overline{1}, \cdots, \overline{j}, \cdots, \overline{S}\}$, which are adopted by two analysts. The mapping (or consolidation scheme) between the two specifications is as follows: $\{\overline{1}\} = \{1\}, \{\overline{2}\} =$

$\{2\}, \cdots, \{\overline{K}\} = \{K\}$, and $\{\overline{S}\} = \{K+1, \cdots, S\}$. Suppose that the current state is the single state $\{\overline{1}\}$ for the second (consolidated) analyst, and $\{1\}$ for the first (original) analyst.

In the first direction of the proof (i.e., proving the sufficient condition), we assume consistent recoveries under both specifications. As a result, the following no-arbitrage conditions on observed price data must be satisfied for the two consistent specifications:

$$A_{\tau+1;1i} = \sum_{j=1}^{S} A_{\tau;1j} A_{ji}, \quad \forall i \in \mathcal{S} \text{ and } \forall \tau \in \{1, 2, 3, \cdots\} \tag{B.1}$$

$$\overline{A}_{\tau+1;\overline{1}\,\overline{i}} = \sum_{\overline{j}=\overline{1}}^{\overline{S}} \overline{A}_{\tau;\overline{1}\,\overline{j}} \overline{A}_{\overline{j}\,\overline{i}}, \quad \forall \overline{i} \in \overline{\mathcal{S}} \text{ and } \forall \tau \in \{1, 2, 3, \cdots\} \tag{B.2}$$

$$\overline{A}_{\tau;\overline{1}\,\overline{i}} = \sum_{j \in \overline{i}} A_{\tau;1j}, \quad \forall \overline{i} \in \overline{\mathcal{S}} \text{ and } \forall \tau \in \{1, 2, 3, \cdots\}. \tag{B.3}$$

The above equation system holds for any horizon τ in the future. However, AD price matrices $\mathbf{A}$ and $\overline{\mathbf{A}}$ contain a fixed number of components to be solved for, resulting in an over-identified equation system. In particular, we substitute (B.3) into (B.2) and obtain

$$\sum_{j \in \overline{i}} A_{\tau+1;1j} = \sum_{\overline{j}=\overline{1}}^{\overline{S}} \left(\sum_{k \in \overline{j}} A_{\tau;1k} \right) \overline{A}_{\overline{j}\,\overline{i}}, \quad \forall \overline{i} \in \overline{\mathcal{S}} \text{ and } \forall \tau \in \{1, 2, 3, \cdots\}. \tag{B.4}$$

The equation system, (B.1) and (B.4), has an infinite number of equations but only $S^2 + (K+1)^2$ unknowns. Hence, in order for the system to have a solution, the components in $\overline{\mathbf{A}}$ must satisfy

$$\overline{A}_{\overline{i}\,\overline{j}} = \sum_{j \in \overline{j}} A_{ij}, \quad \forall \{\overline{i}\} = \{i\} \in \{1, \cdots, K\} \text{ and } \overline{j} \in \overline{\mathcal{S}} \tag{B.5}$$

$$\overline{A}_{\overline{S}\,\overline{j}} = \sum_{j \in \overline{j}} A_{ij}, \quad \forall i \in \overline{S} \text{ and } \overline{j} \in \overline{\mathcal{S}}, \tag{B.6}$$

so that by summing up the equations (B.1) in states $i \in \overline{S}$ we obtain (B.4).

Given that the AD price matrix $\mathbf{A}$ of the original specification $\mathcal{S}$ satisfies (B.5) and (B.6), the right eigenvector, $\mathbf{x}^{(1,R)} = \left[x_1^{(1,R)}, \cdots, x_S^{(1,R)}\right]'$, associated with the largest and positive eigenvalue

δ will satisfy the following form:

$$\begin{bmatrix} A_{11} & \dots & A_{1K} & A_{1,K+1} & \dots & A_{1S} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ A_{K1} & \dots & A_{KK} & A_{K,K+1} & \dots & A_{KS} \\ A_{K+1,1} & \dots & A_{K+1,K} & A_{K+1,K+1} & \dots & A_{K+1,S} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ A_{S1} & \dots & A_{SK} & A_{S,K+1} & \dots & A_{SS} \end{bmatrix} \begin{bmatrix} x_1^{(1,R)} \\ \vdots \\ x_K^{(1,R)} \\ x_{K+1}^{(1,R)} = \overline{x}_{\overline{S}}^{(1,R)} \\ \vdots \\ x_S^{(1,R)} = \overline{x}_{\overline{S}}^{(1,R)} \end{bmatrix} = \delta \begin{bmatrix} x_1^{(1,R)} \\ \vdots \\ x_K^{(1,R)} \\ x_{K+1}^{(1,R)} = \overline{x}_{\overline{S}}^{(1,R)} \\ \vdots \\ x_S^{(1,R)} = \overline{x}_{\overline{S}}^{(1,R)} \end{bmatrix}, \tag{B.7}$$

which implies the marginal utilities satisfy $M_i = M_k$, for all i and k belonging to the same coupled state. By the formula of recovered probabilities (5), it is easy to show that $p_{i\overline{h}} = p_{k\overline{h}}$, $\forall i, k \in \overline{j}$, and $\overline{j}, \overline{h} \in \overline{\mathcal{S}}$.

Next, we prove the other direction (i.e., proving the necessary condition). We assume that the inputs of the original specification satisfy $M_i = M_k, p_{i\overline{h}} = p_{k\overline{h}}$, $\forall i, k \in \overline{j}$, and $\overline{j}, \overline{h} \in \overline{\mathcal{S}}$. According to (5), the original AD price matrix $\mathbf{A}$ satisfy

$$\sum_{j \in \overline{j}} A_{ij} = \sum_{j \in \overline{j}} A_{kj}, \quad \forall i, k \in \overline{S} \text{ and } \overline{j} \in \overline{\mathcal{S}}. \tag{B.8}$$

Since our primary analysis concerns consistency, we take $\mathcal{S}$ as the underlying specification in the current thought experiment. Consequently, we have $\mathbf{A}_{\tau+1} = \mathbf{A}_\tau \mathbf{A}$ holds for all horizon τ. No-arbitrage restriction of trade assets gives (B.3). These conditions together with (B.8) imply that $\overline{\mathbf{A}}_{\tau+1} = \overline{\mathbf{A}}_\tau \overline{\mathbf{A}}$ also holds for all τ and thus we obtain (B.5) and (B.6). Consequently, applying the recovery equation (5), we have

$$\overline{\delta} = \delta; \qquad \frac{\overline{M}_{\overline{j}}}{M_j} = \frac{\overline{M}_{\overline{1}}}{M_1}, \ \forall \{\overline{j}\} = \{j\} \in \{1, \cdots, K\};$$

$$\overline{p}_{t,t+1}(\overline{1}, \overline{j}) = p_{t,t+1}(1, j), \ \forall \{\overline{j}\} = \{j\} \in \{1, \cdots, K\}, \ \forall t; \tag{B.9}$$

$$\overline{p}_{t,t+1}(\overline{1}, \overline{S}) = \sum_{j=K+1}^{S} p_{t,t+1}(1, j), \ \forall t; \qquad \frac{\overline{M}_{\overline{S}}}{\overline{M}_{\overline{1}}} = \sum_{j=K+1}^{S} \frac{p_{t,t+1}(1, j)}{\sum_{i=K+1}^{S} p_{t,t+1}(1, i)} \frac{M_j}{M_1}, \ \forall t.$$

That is, all consistency conditions are satisfied, establishing the recovery consistency for the case of single current state.

Case 2 - Coupled current state: We consider an original specification of $\mathcal{S} = \{1, \cdots, S\}$ and a consolidated specification $\overline{\mathcal{S}} = \{\overline{1}, \overline{K+1}, \cdots, \overline{S-1}, \overline{S}\}$, which are adopted by two analysts. The mapping (or consolidation scheme) between the two specifications is as follows: $\{\overline{1}\} = \{1, \cdots, K\}, \{\overline{K+1}\} = \{K+1\}, \{\overline{K+2}\} = \{K+2\}, \cdots \{\overline{S-1}\} = \{S-1\}$, and $\{\overline{S}\} = \{S\}$. Suppose that the current state is the coupled state $\{\overline{1}\}$ for the second (consolidated) analyst and $\{1\}$ for the first (original) analyst.

In the first direction of the proof (i.e., proving the sufficient condition), we assume consistent recoveries under both specifications. As a result, the same no-arbitrage conditions (B.1)–(B.3) for Case 1 on observed price data must be satisfied for the two consistent specifications. We again obtain the equation system (B.1) and (B.4). In order for the system to have a solution, the components in $\overline{\mathbf{A}}$ must satisfy

$$\overline{A}_{\overline{1}\,\overline{j}} = \sum_{j \in \overline{j}} A_{ij}, \quad \forall i \in \overline{1} \text{ and } \overline{j} \in \overline{\mathcal{S}}, \tag{B.10}$$

$$\overline{A}_{\overline{i}\,\overline{j}} = \sum_{j \in \overline{j}} A_{ij}, \quad \forall \{\overline{i}\} = \{i\} \in \{K+1, \cdots, S\} \text{ and } \overline{j} \in \overline{\mathcal{S}}, \tag{B.11}$$

so that by summing up the equations (B.1) in states $i \in \overline{1}$ we obtain (B.4).

Given that the AD price matrix $\mathbf{A}$ of the original specification $\mathcal{S}$ satisfies (B.10) and (B.11), the right eigenvector, $\mathbf{x}^{(1,R)} = [x_1^{(1,R)}, \cdots, x_S^{(1,R)}]'$, associated with the largest and positive eigenvalue

δ will satisfy the following form:

$$\begin{bmatrix} A_{11} & \dots & A_{1K} & A_{1,K+1} & \dots & A_{1S} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ A_{K1} & \dots & A_{KK} & A_{K,K+1} & \dots & A_{KS} \\ A_{K+1,1} & \dots & A_{K+1,K} & A_{K+1,K+1} & \dots & A_{K+1,S} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ A_{S1} & \dots & A_{SK} & A_{S,K+1} & \dots & A_{SS} \end{bmatrix} \begin{bmatrix} x_1^{(1,R)} = \overline{x}_{\overline{1}}^{(1,R)} \\ \vdots \\ x_K^{(1,R)} = \overline{x}_{\overline{1}}^{(1,R)} \\ x_{K+1}^{(1,R)} \\ \vdots \\ x_S^{(1,R)} \end{bmatrix} = \delta \begin{bmatrix} x_1^{(1,R)} = \overline{x}_{\overline{1}}^{(1,R)} \\ \vdots \\ x_K^{(1,R)} = \overline{x}_{\overline{1}}^{(1,R)} \\ x_{K+1}^{(1,R)} \\ \vdots \\ x_S^{(1,R)} \end{bmatrix} \tag{B.12}$$

which implies the marginal utilities satisfy $M_i = M_k$, for all i and k belonging to the same coupled state. By the formula of recovered probabilities (5), it is easy to show that $p_{i\overline{h}} = p_{k\overline{h}}$, $\forall i, k \in \overline{j}$, and $\overline{j}, \overline{h} \in \overline{\mathcal{S}}$.

Next, we prove the other direction (i.e., proving the necessary condition). We assume that the inputs of the original specification satisfy $M_i = M_k, p_{i\overline{h}} = p_{k\overline{h}}$, $\forall i, k \in \overline{j}$, and $\overline{j}, \overline{h} \in \overline{\mathcal{S}}$. According to (5), the original AD price matrix $\mathbf{A}$ satisfy

$$\sum_{j \in \overline{j}} A_{ij} = \sum_{j \in \overline{j}} A_{kj}, \quad \forall i, k \in \overline{1} \text{ and } \overline{j} \in \overline{\mathcal{S}}. \tag{B.13}$$

Since our primary analysis concerns consistency, we take $\mathcal{S}$ as the underlying specification in the thought experiment. Consequently, we have $\mathbf{A}_{\tau+1} = \mathbf{A}_\tau \mathbf{A}$ holds for all horizon τ. No-arbitrage restriction of traded assets gives (B.3). These conditions together with (B.13) imply that $\overline{\mathbf{A}}_{\tau+1} = \overline{\mathbf{A}}_\tau \overline{\mathbf{A}}$ also holds for all τ and thus we obtain (B.10) and (B.11). Consequently, applying the recovery equation (5), we have

$$\begin{aligned} \overline{\delta} = \delta; \qquad & \overline{p}_{t,t+1}(\overline{1}, \overline{1}) = \sum_{j=1}^{K} p_{t,t+1}(1, j), \ \forall t; \\ \overline{p}_{t,t+1}(\overline{1}, \overline{j}) &= p_{t,t+1}(1, j) \frac{1 - \sum_{i=1}^{K} p_{t,t+1}(1, i)}{1 - \sum_{i=1}^{K} \frac{M_i}{M_1} p_{t,t+1}(1, i)}, \ \forall \{\overline{j}\} = \{j\} \in \{K+1, \cdots, S\}, \ \forall t; \\ \frac{\overline{M}_{\overline{j}}}{\overline{M}_{\overline{1}}} &= \frac{M_j}{M_1} \frac{1 - \sum_{i=1}^{K} p_{t,t+1}(1, i)}{1 - \sum_{i=1}^{K} \frac{M_i}{M_1} p_{t,t+1}(1, i)}, \ \forall \{\overline{j}\} = \{j\} \in \{K+1, \cdots, S\}, \ \forall t. \end{aligned} \tag{B.14}$$

That is, all consistency conditions are satisfied, implying the recovery consistency for the case of coupled current state. Together with the derivation above for the case of single current state, this establishes Proposition 1

B.2 Best-Fit Recovery

Given a specification $\mathcal{S}$ of S states, the basic recovery approach employs price data of just enough $S+1$ tenors τ of long-term AD assets to solve for the $S \times S$ one-period AD price matrix $\mathbf{A}$ from $\mathbf{A}_{\tau+1} = \mathbf{A}_\tau \, \mathbf{A}$ (6). Another analyst perceiving another specification $\overline{\mathcal{S}}$ of $\overline{S} < S$ states needs less price data (of $\overline{S}+1$ tenors) to imply the $\overline{S} \times \overline{S}$ AD price matrix $\overline{\mathbf{A}}$ from $\overline{\mathbf{A}}_{\tau+1} = \overline{\mathbf{A}}_\tau \, \overline{\mathbf{A}}$ (8). The inadvertent (and relative) loss of price information employed in the recovery based on a coarser $\overline{\mathcal{S}}$ gives rise to the recovery consistency issue (Section 3).

A possible approach to mitigate the consistency issue is to employ more price data than needed to obtain best-fit (least squares) recovery results in different specifications that might exhibit no mutual inconsistencies. The best-fit recovery approach transforms and interprets the recovery systems $\mathbf{A}$ from $\mathbf{A}_{\tau+1} = \mathbf{A}_\tau \, \mathbf{A}$ and $\overline{\mathbf{A}}_{\tau+1} = \overline{\mathbf{A}}_\tau \, \overline{\mathbf{A}}$ as regression equation systems, employing flexibly all T available price data inputs (possibly $T > S$, $\overline{S}$) to uniquely obtain a set of least-squares characteristics (Section 4). The best-fit regressions based on these recovery equation systems produce the respective one-period AD asset price matrices

$$\begin{aligned} \text{Original system:} \quad & \mathbf{A} = \left[\mathbf{A}_T' \, \mathbf{A}_T\right]^{-1} \mathbf{A}_T' \, \mathbf{A}_{T+1}, \\ \text{Consolidated system:} \quad & \overline{\mathbf{A}} = \left[\overline{\mathbf{A}}_T' \, \overline{\mathbf{A}}_T\right]^{-1} \overline{\mathbf{A}}_T' \, \overline{\mathbf{A}}_{T+1}, \end{aligned} \tag{B.15}$$

where matrices $\mathbf{A}_T$ and $\overline{\mathbf{A}}_T$ contain all available original and consolidated long-term AD asset prices associated respectively with $\mathcal{S}$ and $\overline{\mathcal{S}}$. Employing more price data (tenors) does not weaken the consistency requirement and hence does not alleviate the consistency issue in the recovery approach. Intuitively, the law of one price that relate asset prices in $\mathbf{A}_T$ and $\overline{\mathbf{A}}_T$ applies uniformly across various tenors for all price data points.[5] Hence, adding tenors does not resolve the issue. Technically,

[5] Given the same initial current state i, the current AD asset prices observed by the two analysts are related by the law of one price $\overline{A}_{\tau;i\,\overline{j}} = \sum_{j \in \overline{j}} A_{\tau;ij}$, $\forall \tau \in \{1, \ldots, T\}$.

to see this in the perturbative setup, consider the perturbative expansions $\mathbf{A}_T(\varepsilon) = \mathbf{A}_{T0} + \varepsilon\,\mathbf{B}_T$ and $\overline{\mathbf{A}}_T(\varepsilon) = \overline{\mathbf{A}}_{T0} + \varepsilon\,\overline{\mathbf{B}}_T$. Substituting these expansions into the best-fit solutions (B.15) yields the perturbative expressions for the one-period AD price matrices (using all available price data) that are subject to the same issue analyzed in Section 3 (using just enough price data). That is, as long as specifications $\mathcal{S}$ and $\overline{\mathcal{S}}$ are not consistent, the perturbative components of these one-period AD price matrices (using all available price data) are not consistent, leading to inconsistent eigenspaces and inconsistent recovery results that these eigenspaces represent. In other words, the relative loss of information between two subjective recovery implementations persists as they are associated with inconsistent specifications.

B.2.1 Recursive AD Pricing Equations and Counting Arguments

This appendix elaborates on the required numbers of traded assets and their tenors, whose prices are observed, to solve for the implied one-period AD price matrix in the Markovian setting of the Recovery Theorem. We recall the convention that 1 denotes the current state of a state space of S states, and $A_{\tau;ij}$ denotes the price of the τ-period AD asset initiated in state i and offering a unit payoff in state j in τ periods (and nothing else). In this convention, the first row of the $S \times S$ one-period AD price matrix $\mathbf{A}$ contains the prices of the S currently traded (i.e., observed) AD assets that pay off next period, $A_{1j} = A_{1;1j}$, $j \in \{1, \ldots, S\}$. That is, only the remaining $S - 1$ rows of $\mathbf{A}$ need to be solved. As a result, the recursive system (of S^2 pricing equations) in the $S \times S$ matrix equation $\mathbf{A}_{\tau+1} = \mathbf{A}_\tau\ \mathbf{A}$ (6) can be reduced to solving just for $S - 1$ unobserved rows of $\mathbf{A}$. To obtain this reduction, we drop the last row (i.e., S-th row) of matrices $\mathbf{A}_{\tau+1}$ and $\mathbf{A}_\tau$ of observed AD prices in Equation (6), turning it into

$$\begin{bmatrix} A_{2;11} & \ldots & A_{2;1S} \\ \vdots & \ddots & \vdots \\ A_{S;11} & \ldots & A_{S;1S} \end{bmatrix} = \begin{bmatrix} A_{1;11} & \ldots & A_{1;1S} \\ \vdots & \ddots & \vdots \\ A_{S-1;11} & \ldots & A_{S-1;1S} \end{bmatrix} \times \underbrace{\begin{bmatrix} A_{11} & \ldots & A_{1S} \\ \vdots & \ddots & \vdots \\ A_{S1} & \ldots & A_{SS} \end{bmatrix}}_{\mathbf{A}},$$

$$
= \underbrace{\begin{bmatrix} A_{1;11} \\ \vdots \\ A_{S-1;11} \end{bmatrix}}_{\equiv \mathbf{A}_{\tau;:1}} \times \underbrace{\begin{bmatrix} A_{11} & \dots & A_{1S} \end{bmatrix}}_{\equiv \mathbf{A}_{1:}} + \begin{bmatrix} A_{1;12} & \dots & A_{1;1S} \\ \vdots & \ddots & \vdots \\ A_{S-1;12} & \dots & A_{S-1;1S} \end{bmatrix} \times \underbrace{\begin{bmatrix} A_{21} & \dots & A_{2S} \\ \vdots & \ddots & \vdots \\ A_{S1} & \dots & A_{SS} \end{bmatrix}}_{\mathbf{A}_{\text{reduced}}}, \tag{B.16}
$$

where $(S-1) \times S$ matrix $\mathbf{A}_{\text{reduced}}$ contains the last $S-1$ rows of $\mathbf{A}$, which we need to solve for. Compared to the system (6) which involves $S+1$ tenors ($\tau \in \{1, \dots, S+1\}$), the reduced system (B.16) employs AD assets of only S tenors ($\tau \in \{1, \dots, S\}$). The reduced one-period AD matrix $\mathbf{A}_{\text{reduced}}$ is given by

$$
\mathbf{A}_{\text{reduced}} = \begin{bmatrix} A_{1;12} & \dots & A_{1;1S} \\ \vdots & \ddots & \vdots \\ A_{S-1;12} & \dots & A_{S-1;1S} \end{bmatrix}^{-1} \times \left(\begin{bmatrix} A_{2;11} & \dots & A_{2;1S} \\ \vdots & \ddots & \vdots \\ A_{S;11} & \dots & A_{S;1S} \end{bmatrix} - \mathbf{A}_{\tau;:1} \times \mathbf{A}_{1:} \right), \tag{B.17}
$$

where $(S-1) \times 1$ matrix $\mathbf{A}_{\tau;:1}$ and $1 \times S$ matrix $\mathbf{A}_{1:}$ have been defined in (B.16). Compared to the solution of the full one-period AD price matrix $\mathbf{A} = \mathbf{A}_{\tau}^{-1} \mathbf{A}_{\tau+1}$ (6), the advantage of solution (B.17) of the reduced one-period AD price matrix $\mathbf{A}_{\text{reduced}}$ is the employment of one fewer tenor as mentioned earlier. The disadvantage is that the solution (B.17) is less concise algebraically, separating the first and the remaining rows of $\mathbf{A}$. Since the consistency of recoveries under different specifications is determined by the relation between these specifications (Definition 1) regardless of whether we use solution (6) or (B.17) for the AD price matrix, we opt for the former in the main text to ease our notation and exposition. In practice, our empirical analysis in Section 4 employs options of all tenors in our sample in a least-squares implementation of the recovery.

B.3 Recovery Inconsistencies: A Further Quantitative Analysis

This appendix presents a further quantitative elaboration on the recovery consistency issue. We analyze the recovery inconsistencies by deriving explicit expressions for the inverses of AD price matrices in Appendix B.3.1. We discuss the spectral gaps (eigenvalue distributions) of the implied one-period AD price matrices and their relationships to recovery inconsistencies in Appendix B.3.2.

B.3.1 Inverses of AD Price Matrices and Recovery Inconsistencies

We first relate AD price matrices to Vandermonde matrix, whose inverse is known analytically, before presenting a further quantitative analysis of recovery inconsistencies. Recall from Section 3 that the recovery inconsistencies originate from the inconsistency of the perturbative component $\overline{\mathbf{B}}$ of the one-period AD price matrix. As a result, a quantitative analysis of the recovery inconsistencies relies on an explicit inversion of the price matrix $\overline{\mathbf{A}}_{0\tau}^{-1}$ (Case 2, Proposition 2). In fact, $\overline{\mathbf{A}}_{0\tau}^{-1}$ and $\overline{\mathbf{B}}$ feature predominantly in the divergence of the recovered time (21) and risk (23) preferences.

To obtain $\overline{\mathbf{A}}_{0\tau}^{-1}$ explicitly, we first express every row $\mathbf{A}_{0\tau i,:}$ of $\mathbf{A}_{0\tau}$ (resp., every row $\overline{\mathbf{A}}_{0\tau\bar{i},:}$ of $\overline{\mathbf{A}}_{0\tau}$) as a linear combination of the unperturbed left eigenvectors $\{\mathbf{x}_0^{(k,L)}\}$, $k \in \{1, \ldots, S\}$ (resp., $\{\overline{\mathbf{x}}_0^{(\bar{k},L)}\}$, $\bar{k} \in \{1, \ldots, \overline{S}\}$), because these eigenvectors span the space of $1 \times S$ (resp., $1 \times \overline{S}$) row vectors,

$$\begin{cases} \mathbf{A}_{0\tau i,:} = \sum_{k=1}^{S} \gamma_{ik}\, \mathbf{x}^{(k,L)}, \\ \text{or in matrix form } \mathbf{A}_{0\tau} = \mathbf{\Gamma}\, \mathbf{X}^{L}, \end{cases} \qquad \begin{cases} \overline{\mathbf{A}}_{0\tau\bar{i},:} = \sum_{\bar{k}=\bar{1}}^{\overline{S}} \overline{\gamma}_{\bar{i}\,\bar{k}}\, \overline{\mathbf{x}}^{(\bar{k},L)}, \\ \text{or in matrix form } \overline{\mathbf{A}}_{0\tau} = \overline{\mathbf{\Gamma}}\, \overline{\mathbf{X}}^{L}, \end{cases} \tag{B.18}$$

where $\mathbf{\Gamma}$ and $\overline{\mathbf{\Gamma}}$ are matrices of respective coefficients $\{\gamma_{ik}\}$ and $\{\overline{\gamma}_{\bar{i}\,\bar{k}}\}$. Since right and left eigenvectors are orthonormal, $\mathbf{X}_0^L \mathbf{X}_0^R = \mathbb{1}_{S\times S}$ and $\overline{\mathbf{X}}_0^L \overline{\mathbf{X}}_0^R = \mathbb{1}_{\overline{S}\times\overline{S}}$, the coefficient matrices are $\mathbf{\Gamma} = \mathbf{A}_{0\tau}\mathbf{X}^R$ and $\overline{\mathbf{\Gamma}} = \overline{\mathbf{A}}_{0\tau}\overline{\mathbf{X}}^R$. Using the recursive construction (6) of the unperturbed τ-period AD matrices $\mathbf{A}_{0\tau} = \mathbf{A}_{0\tau-1}\mathbf{A}_0$ and $\overline{\mathbf{A}}_{0\tau} = \overline{\mathbf{A}}_{0\tau-1}\overline{\mathbf{A}}_0$, and the fact that columns of matrices $\mathbf{X}_0^R$ and $\overline{\mathbf{X}}_0^R$ are right eigenvalues of one-period AD price matrices $\mathbf{A}_0$ and $\overline{\mathbf{A}}_0$, we obtain explicit expressions for the coefficient matrices[6]

$$
\mathbf{\Gamma} = \underbrace{\begin{bmatrix} \delta_0^{(1)} & \dots & \delta_0^{(S)} \\ \vdots & \ddots & \vdots \\ \left[\delta_0^{(1)}\right]^S & \dots & \left[\delta_0^{(S)}\right]^S \end{bmatrix}}_{\equiv \mathbf{D}_S} \underbrace{\begin{bmatrix} x_{01}^{(1,R)} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & x_{01}^{(S,R)} \end{bmatrix}}_{\equiv \mathbf{Diag}\left(X_{01}^{(R)}\right)} = \underbrace{\mathbf{D}_S}_{S\times S} \underbrace{\mathbf{Diag}\left(X_{01}^{(R)}\right)}_{S\times S},
$$

$$
\overline{\mathbf{\Gamma}} = \underbrace{\begin{bmatrix} \delta_0^{(1)} & \dots & \delta_0^{(\overline{S})} \\ \vdots & \ddots & \vdots \\ \left[\delta_0^{(1)}\right]^{\overline{S}} & \dots & \left[\delta_0^{(\overline{S})}\right]^{\overline{S}} \end{bmatrix}}_{\equiv \overline{\mathbf{D}}_{\overline{S}}} \underbrace{\begin{bmatrix} \overline{x}_{01}^{(1,R)} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \overline{x}_{01}^{(\overline{S},R)} \end{bmatrix}}_{\equiv \mathbf{Diag}\left(\overline{X}_{01}^{(R)}\right)} = \underbrace{\overline{\mathbf{D}}_{\overline{S}}}_{\overline{S}\times \overline{S}} \underbrace{\mathbf{Diag}\left(\overline{X}_{01}^{(R)}\right)}_{\overline{S}\times \overline{S}}, \tag{B.19}
$$

where diagonal matrix $\mathbf{Diag}(X_{01}^{(R)})$ (resp., $\mathbf{Diag}(\overline{X}_{01}^{(R)})$) contains the first component of the unperturbed right eigenvectors $\mathbf{x}_0^{(k,R)}$, $k \in \{1,\dots,S\}$ (resp., $\overline{\mathbf{x}}_0^{(\overline{k},R)}$, $\overline{k} \in \{1,\dots,\overline{S}\}$). The k-th row of matrix $\mathbf{D}_S$ (resp., $\overline{\mathbf{D}}_{\overline{S}}$) as defined in the above expressions contains the k-th exponent of the eigenvalues $\{\delta_0^{(1)},\dots,\delta_0^{(S)}\}$ (resp., $\{\delta_0^{(1)},\dots,\delta_0^{(\overline{S})}\}$). Therefore, $\mathbf{D}_S$ and $\overline{\mathbf{D}}_{\overline{S}}$ are Vandermonde matrices (see the digression (B.25) and (B.26) below), whose inverses, $\mathbf{D}_S^{-1}$ and $\overline{\mathbf{D}}_{\overline{S}}^{-1}$, have known closed-form expressions. Specifically, the element $(1,i)$ of the inverse Vandermonde matrix $\left[\overline{\mathbf{D}}_{\overline{S}}^{-1}\right]$ is (e.g., Man (2017))

$$
\left[\overline{\mathbf{D}}_{\overline{S}}^{-1}\right]_{1i} = \frac{\left(\delta_0^{(1)}\right)^{\overline{S}-i} + a_1\left(\delta_0^{(1)}\right)^{\overline{S}-i-1} + \dots + a_{\overline{S}-i-1}\left(\delta_0^{(1)}\right) + a_{\overline{S}-i}}{\delta_0^{(1)}\prod_{j\neq 1}^{\overline{S}}(\delta_0^{(1)} - \delta_0^{(j)})}, \quad i \in \{1,\dots,\overline{S}\}, \tag{B.20}
$$

$$
a_0 = 1, \quad a_1 = -\sum_j^{\overline{S}} \delta_0^{(j)}, \quad a_2 = \sum_{j\neq m}^{\overline{S}} \delta_0^{(j)}\,\delta_0^{(m)}, \quad \dots, \quad a_{\overline{S}-1} = (-1)^{\overline{S}-1}\prod_j^{\overline{S}}\delta_0^{(j)}, \quad a_h = 0, \forall h < 0.
$$

Substituting the expressions (B.19) of $\mathbf{\Gamma}$ and $\overline{\mathbf{\Gamma}}$ into (B.18) yields the inverse matrices $\overline{\mathbf{A}}_{0\tau}^{-1} = \overline{\mathbf{X}}^R\,\overline{\mathbf{\Gamma}}^{-1} = \overline{\mathbf{X}}^R\mathbf{Diag}\left(\frac{1}{\overline{X}_{01}^{(R)}}\right)\overline{\mathbf{D}}_{\overline{S}}^{-1}$ and $\mathbf{A}_{0\tau-} = \mathbf{\Gamma}_-\mathbf{X}^L = \mathbf{D}_{S-}\,\mathbf{Diag}\left(X_{01}^{(R)}\right)\mathbf{X}^L$, which transform

[6] Recall that the dominant unperturbed eigenspaces are consistent, hence $\delta_0^{(1)} = \overline{\delta}_0^{(1)} = \delta_0$ as noted below (20). To arrive at these expressions, recall the convention that the current state is 1. As a result, the first row of observable price matrix $\mathbf{A}_{0\tau}$ (6) contains the current one-period AD prices, $\mathbf{A}_{0\tau;1:} = \mathbf{A}_{01:}$, the second row $\mathbf{A}_{0\tau;2:} = \mathbf{A}_{01:}\,\mathbf{A}_0$, and so on to the S-th row $\mathbf{A}_{0\tau;S:} = \mathbf{A}_{01:}\,\mathbf{A}_0^{S-1}$. The product $\mathbf{A}_{0\tau}\,\mathbf{X}_0^R$ then is composed of various rows of the form $\mathbf{A}_{01:}\,\mathbf{A}_0^k\,\mathbf{X}_0^R$, $k \in \{0,\dots,S-1\}$, resulting in (B.19).

the divergence (21) in analysts' recovered time preferences (in the first order of ε) into

$$\overline{\delta}^{(1)}(\varepsilon) - \delta^{(1)}(\varepsilon) \;\sim\; \left[\overline{\mathbf{x}}_0^{(1,L)}\;\overline{\mathbf{A}}_{0\tau}^{-1}\mathbf{A}_{0\tau-} \;-\; \mathbf{x}_0^{(1,L)}\right]\;\mathbf{B}^+\;\overline{\mathbf{x}}_0^{(1,R)}$$

$$= \left[\overline{\mathbf{x}}_0^{(1,L)}\;\overline{\mathbf{X}}^R\,\mathbf{Diag}\left(\frac{1}{\overline{X}_{01}^{(R)}}\right)\,\overline{\mathbf{D}}_{\overline{S}}^{-1}\,\mathbf{D}_{S-}\,\mathbf{Diag}\left(X_{01}^{(R)}\right)\;\mathbf{X}^L \;-\; \mathbf{x}_0^{(1,L)}\right]\;\mathbf{B}^+\;\overline{\mathbf{x}}_0^{(1,R)}. \tag{B.21}$$

Note that the $\overline{S}$ eigenvalues $\{\delta_0^{(k)}\}, \in \{1,\ldots,\overline{S}\}$, in matrix $\overline{\mathbf{D}}_{\overline{S}}$ (B.18) are also in matrix $\mathbf{D}_S$, and the $\overline{S}$ first components $\{\overline{x}_{01}^{(k,R)}\}$, $k \in \{1,\ldots,\overline{S}\}$ in matrix $\mathbf{Diag}(\overline{X}_{01}^{(R)})$ are also in matrix $\mathbf{Diag}(X_{01}^{(R)})$ (per relationships (13) of Remark 1). The cancellation of these quantities in (B.21) yields an explicit decomposition for the divergence in the recovered time preferences[7]

$$\overline{\delta}^{(1)}(\varepsilon) - \delta^{(1)}(\varepsilon) \;\sim\; \sum_{k=\overline{S}+1}^{S} \underbrace{\left(\sum_{i=1}^{\overline{S}} \left[\overline{\mathbf{D}}_{\overline{S}}^{-1}\right]_{1i}\;\left[\delta_0^{(k)}\right]^i\;\frac{x_{01}^{(k,R)}}{x_{01}^{(1,R)}}\right)}_{\equiv C_{1k}}\;\mathbf{x}_0^{(k,L)}\;\mathbf{B}^+\;\overline{\mathbf{x}}_0^{(1,R)}$$

$$= \sum_{k=\overline{S}+1}^{S} C_{1k}\;\mathbf{x}_0^{(k,L)}\;\mathbf{B}^+\;\overline{\mathbf{x}}_0^{(1,R)}, \tag{B.22}$$

where $\left[\overline{\mathbf{D}}_{\overline{S}}^{-1}\right]_{1i}$ is the element $(1,i)$ of the inverse Vandermonde matrix $\left[\overline{\mathbf{D}}_{\overline{S}}^{-1}\right]$ (B.20).

Compared to the qualitative expression (21), the explicit decomposition (B.22) offers deeper quantitative insights into factors driving the inconsistencies in the recovered time preferences. First, only the couplings between the dominant eigenvector $\overline{\mathbf{x}}_0^{(1,R)}$ and extra eigenvectors $\mathbf{x}_0^{(k,L)}$, $k \in \{\overline{S}+1,\ldots,S\}$, contribute to the inconsistencies. This is because the couplings between the dominant and the lower eigenvectors $\mathbf{x}_0^{(k,L)}$, $k \in \{1,\ldots,\overline{S}\}$, are present and common in the recovery results for both original ($\mathcal{S}$) and consolidated ($\overline{\mathcal{S}}$) specifications, hence are canceled out in the divergence of the two recovery results (Footnote 7). In the special case of consistent perturbative components (17), $\mathbf{x}_0^{(k,L)}\;\mathbf{B}^+ = 0$ for every (extra) state $k \in \{\overline{S}+1,\ldots,S\}$ (Case 2, Proposition 2), the couplings with the extra eigenvectors $\mathbf{x}_0^{(k,L)}$ vanish and the recovery consistency is preserved, $\overline{\delta}^{(1)}(\varepsilon) = \delta^{(1)}(\varepsilon)$.

[7] Because the $\overline{S}\times S$ matrix $\mathbf{D}_{S-}$ is obtained by dropping $(S-\overline{S})$ extra rows (not needed for the recovery in $\overline{\mathcal{S}}$) of the $S\times S$ matrix $\mathbf{D}_S$, the matrix product $\overline{\mathbf{D}}_{\overline{S}}^{-1}\,\mathbf{D}_{S-}$ can be separated into two blocks. The left block is the identity matrix $\mathbb{1}_{\overline{S}\times\overline{S}}$ that helps to cancel term $\mathbf{x}_0^{(1,L)}$ in (B.21). The remaining (right) block retains the last $(S-\overline{S})$ columns of matrix $\mathbf{D}_S$. Further, the orthonormality between right and left eigenvectors implies $\overline{\mathbf{x}}_0^{(1,L)}\,\overline{\mathbf{X}}^R = (1,0,\ldots,0)$ which retains only the first row of $\overline{\mathbf{D}}_{\overline{S}}^{-1}$ in the resulting divergence (B.22).

Second, the couplings between the dominant eigenvector and extra eigenvectors $\{\mathbf{x}_0^{(k,L)}\}$ are scaled by corresponding factors $\{C_{1k}\}$, $k \in \{\overline{S}+1,\ldots,S\}$ in the divergence (B.22). Being linear combinations of the elements (B.20) of the inverse Vandermonde matrix, these factors are explicit and shed light on how recovery inconsistencies vary with analyst's subjective specification $\overline{\mathcal{S}}$. Elements $\left\{\left[\overline{\mathbf{D}}_{\overline{S}}^{-1}\right]_{1i}\right\}$, $i \in \{1,\ldots,\overline{S}\}$, are rational functions of the eigenvalues $\{\delta_0^{(j)}\}$, $j \in \{1,\ldots,\overline{S}\}$, so are the scaling factors $\{C_{1k}\}$, $k \in \{\overline{S}+1,\ldots,S\}$. For a finer subjective specification $\overline{\mathcal{S}}$, i.e. a larger number of coupled states $\overline{S}$, the denominators of $\overline{\mathbf{D}}_{\overline{S}}^{-1}$'s (and the scaling factors) are polynomials of higher orders in the eigenvalues $\prod_{j\neq 1}^{\overline{S}}(\delta_0^{(1)}-\delta_0^{(j)})$. As a result, when the eigenvalues of the underlying one-period AD price matrix $\mathbf{A}_0$ are distributed closely (i.e., a dense spectrum or small spectral gap), a finer specification $\overline{\mathcal{S}}$ may be associated with larger scaling factors (i.e., rational functions with larger numbers of poles). We note that a partial fraction decomposition does not mitigate the issue that the magnitudes of the scaling factors, or the degree and number of poles of the denominators of $\overline{\mathbf{D}}_{\overline{S}}^{-1}$'s and $\{C_{1k}\}$, increase with a finer $\overline{\mathcal{S}}$ (a larger $\overline{S}$) when the eigenvalues are similar. To see this in an analogous example, note that the partial fraction decomposition generates the identity,

$$\frac{1}{\left(x-\delta_0^{(2)}\right)\left(x-\delta_0^{(3)}\right)} = \frac{a}{x-\delta_0^{(2)}}+\frac{b}{x-\delta_0^{(3)}}, \quad \text{with:} \quad a=\frac{1}{\delta^{(3)}-\delta^{(2)}}, \quad b=\frac{1}{\delta^{(2)}-\delta^{(3)}}. \tag{B.23}$$

When $x=\delta_0^{(1)}$ in this example, the identity becomes

$$\frac{1}{\left(\delta_0^{(1)}-\delta_0^{(2)}\right)\left(\delta_0^{(1)}-\delta_0^{(3)}\right)} = \frac{1}{\left(\delta_0^{(3)}-\delta_0^{(2)}\right)\left(\delta_0^{(1)}-\delta_0^{(2)}\right)}+\frac{1}{\left(\delta_0^{(2)}-\delta_0^{(3)}\right)\left(\delta_0^{(1)}-\delta_0^{(3)}\right)}. \tag{B.24}$$

That is, all terms on both sides of this equality are of the same order, each has two poles. In the general case of inconsistent perturbative components, the couplings $\mathbf{x}_0^{(k,L)}\ \mathbf{B}^+\ \overline{\mathbf{x}}_0^{(1,R)}$, $k \in \{\overline{S}+1,\ldots,S\}$, are non-zero (and unconstrained by the consistency requirement). When scaled by larger factors $\{C_{1k}\}$ associated with a finer subjective specification $\overline{\mathcal{S}}$, these couplings may produce a larger divergence (B.22), i.e., larger recovery inconsistencies.

A comparison of the divergences in the recovered risk preferences' loadings (23) and in the recovered time preferences (21) indicates that an explicit expression for the former can be obtained in an identical procedure, i.e., replacing C_{1k} in (B.22) by C_{hk}, and $\left[\overline{\mathbf{D}}_{\overline{S}}^{-1}\right]_{1i}$ in (B.20) by $\left[\overline{\mathbf{D}}_{\overline{S}}^{-1}\right]_{hi}$, where $h \in \{2, \ldots, \overline{S}\}$. The resulting explicit expressions for the recovered risk preferences then implicate similar quantitative findings on the sources driving the inconsistencies in the recovered marginal utilities.

In summary, the analysts' recovery results remain divergent as long as their recovery specifications remain inconsistent with each other. The perturbative setup demonstrates that the divergence between an analyst's recovery results and the underlying is scaled by rational functions of the AD price matrix's eigenvalues. The analyst's finer subjective specification suppresses the degree of these rational functions, which is the difference between the degrees of the polynomials in the numerator and denominator of a rational function. As a result, when the eigenvalues are distributed closely, a finer subjective specification may actually increase the divergence, hence the inconsistencies, of the recovery results. This is because a dense spectrum of the AD price matrix implies that higher eigenspaces perturb and distort the dominant (recovery) one more strongly.

A digression on the Vandermonde matrix: To relate the Vandermonde matrix to the recovery setting, we recall a well-known application of this matrix (and its inverse). This application concerns the exact fitting of a $(n-1)-$degree polynomial $f(x)$ that passes through n given points $\{(x_0, y_0) \ldots (x_{n-1}, y_{n-1})\}$. The fitting "recovers" n unknown coefficients $\{a_0, \ldots, a_{n-1}\}$ of polynomial $f(x)$ via an equation system

$$\text{Vandermonde system:} \quad \begin{cases} a_0 + a_1 x_0 + a_2 x_0^2 + \ldots + a_{n_1} x_0^{n-1} & = \quad y_0, \\ a_0 + a_1 x_1 + a_2 x_1^2 + \ldots + a_{n_1} x_1^{n-1} & = \quad y_1, \\ \vdots & \\ a_0 + a_1 x_{n-1} + a_2 x_{n-1}^2 + \ldots + a_{n_1} x_{n-1}^{n-1} & = \quad y_{n-1}, \end{cases} \tag{B.25}$$

or in matrix form

$$[a_0 \;\; a_1 \;\; \dots \;\; a_{n-1}] \underbrace{\begin{bmatrix} 1 & 1 & \dots & 1 \\ x_0 & x_1 & \dots & x_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_0^{n-1} & x_1^{n-1} & \dots & x_{n-1}^{n-1} \end{bmatrix}}_{\text{Vandermonde matrix}} = [y_0 \;\; y_1 \;\; \dots \;\; y_{n-1}]\,. \tag{B.26}$$

From this follows a unique solution for the coefficients, $\{a_k\}$, $k \in \{0, \dots, n-1\}$ from x's and y's coordinates.

A change of notation relates this equation system to our recovery setting (B.19). After replacing S with n, and $\delta_0^{(k+1)}$ by x_k (for $k \in \{0, \dots, S-1\}$), the matrix product $\mathbf{D}_S\,\mathbf{Diag}\left(\frac{1}{\delta_0^{(1)}}, \frac{1}{\delta_0^{(2)}}, \dots, \frac{1}{\delta_0^{(S)}}\right)$ (where $S \times S$ matrix $\mathbf{D}_S$ is given (B.19)) becomes the $n \times n$ Vandermonde matrix on the LHS of the system (B.26). Since the analytical expression for the inverse of the Vandermonde matrix is known, the analytical expression for the inverse matrix $[\mathbf{D}_S]^{-1}$ is also known. The same holds for $\overline{\mathbf{D}}_{\overline{S}}$ in (B.19), which yields explicit expressions for its elements $\left[\overline{\mathbf{D}}_{\overline{S}}^{-1}\right]_{1i}$ (B.20).

B.3.2 Spectral Gaps of AD Price Matrices

Consider the eigenproblem of AD price matrices $\mathbf{A}(\varepsilon)$ and $\overline{\mathbf{A}}(\varepsilon)$ (16) in the perturbative setup,

$$\mathbf{A}(\varepsilon)\,\mathbf{x}^{(k,R)}(\varepsilon) = \delta^{(k)}(\varepsilon)\,\mathbf{x}^{(k,R)}(\varepsilon), \text{ with } \begin{cases} \mathbf{A}(\varepsilon) = \mathbf{A}_0 + \varepsilon\mathbf{B} \\ \delta^{(k)}(\varepsilon) = \delta_0^{(k)} + \varepsilon\Delta\delta^{(k)}, & k \in \{1, \dots, S\}, \\ \mathbf{x}^{(k,R)}(\varepsilon) = \mathbf{x}_0^{(k,R)} + \varepsilon\Delta\mathbf{x}^{(k,R)}, \end{cases} \tag{B.27}$$

and

$$\overline{\mathbf{A}}(\varepsilon)\,\overline{\mathbf{x}}^{(k,R)}(\varepsilon) = \overline{\delta}^{(k)}(\varepsilon)\,\overline{\mathbf{x}}^{(k,R)}(\varepsilon), \text{ with } \begin{cases} \overline{\mathbf{A}}(\varepsilon) = \overline{\mathbf{A}}_0 + \varepsilon\overline{\mathbf{B}} \\ \overline{\delta}^{(k)}(\varepsilon) = \overline{\delta}_0^{(k)} + \varepsilon\Delta\overline{\delta}^{(k)}, & k \in \{1, \dots, \overline{S}\}, \\ \overline{\mathbf{x}}^{(k)}(\varepsilon) = \overline{\mathbf{x}}_0^{(k)} + \varepsilon\Delta\overline{\mathbf{x}}^{(k)}, \end{cases} \tag{B.28}$$

Substituting the perturbative expansions in the second part into the first part of (B.27), matching terms linear in ε, multiplying to the left of the resulting equation by the unperturbed k-th left (row) eigenvector $\mathbf{x}_0^{(k,L)}$, and employing the orthonormality between left and right eigenvectors, yield the eigenvalues (i.e., spectrum) of the AD matrix $\mathbf{A}(\varepsilon)$ associated with the original specification (and similarly for the spectrum of the AD matrix $\overline{\mathbf{A}}(\varepsilon)$),

$$\begin{cases} \delta^{(k)}(\varepsilon) = \delta_0^{(k)} + \varepsilon\, \mathbf{x}_0^{(k,L)}\, \mathbf{B}\, \mathbf{x}_0^{(k,R)}, \\ k \in \{1, \dots, S\}, \end{cases} \qquad \begin{cases} \overline{\delta}^{(k)}(\varepsilon) = \overline{\delta}_0^{(k)} + \varepsilon\, \overline{\mathbf{x}}_0^{(k,L)}\, \overline{\mathbf{B}}\, \overline{\mathbf{x}}_0^{(k,R)}, \\ k \in \{1, \dots, \overline{S}\}. \end{cases} \tag{B.29}$$

Since the unperturbed components of the perturbative setup (14) are consistent, an application of Remark 1 (Equation (13)) implies that the two spectra share $\overline{S}$ unperturbed components of their eigenvalues

$$\delta_0^{(k)} = \overline{\delta}_0^{(k)}, \qquad \forall k \in \{1, \dots, \overline{S}\}. \tag{B.30}$$

The distances from the k-th eigenvalue to the dominant (first) eigenvalue (or, spectral gaps) in the original and consolidated spectra are

$$\delta^{(k)}(\varepsilon) - \delta^{(1)}(\varepsilon) = \left[\delta_0^{(k)} - \delta_0^{(1)}\right] \;+\; \varepsilon \left[\mathbf{x}_0^{(k,L)}\, \mathbf{B}\, \mathbf{x}_0^{(k,R)} - \mathbf{x}_0^{(1,L)}\, \mathbf{B}\, \mathbf{x}_0^{(1,R)}\right], \quad \forall k \in \{1, \dots, S\},$$

$$\overline{\delta}^{(k)}(\varepsilon) - \overline{\delta}^{(1)}(\varepsilon) = \left[\overline{\delta}_0^{(k)} - \overline{\delta}_0^{(1)}\right] \;+\; \varepsilon \left[\overline{\mathbf{x}}_0^{(k,L)}\, \overline{\mathbf{B}}\, \overline{\mathbf{x}}_0^{(k,R)} - \overline{\mathbf{x}}_0^{(1,L)}\, \overline{\mathbf{B}}\, \overline{\mathbf{x}}_0^{(1,R)}\right], \quad \forall k \in \{1, \dots, \overline{S}\}.$$

Since these spectral gaps share identical unperturbed components (B.30) for $k \in \{1, \dots, \overline{S}\}$, their relative spectral gaps for these eigenvalues are linear in ε. Up to the multiplicative constant of ε, the relative gaps are[8]

$$\begin{aligned} RG^{(k)} &\equiv \left[\overline{\delta}^{(k)}(\varepsilon) - \overline{\delta}^{(1)}(\varepsilon)\right] - \left[\delta^{(k)}(\varepsilon) - \delta^{(1)}(\varepsilon)\right] \\ &\sim \left(\overline{\mathbf{x}}_0^{(k,L)}\, \overline{\mathbf{B}}\, \overline{\mathbf{x}}_0^{(k,R)} - \overline{\mathbf{x}}_0^{(1,L)}\, \overline{\mathbf{B}}\, \overline{\mathbf{x}}_0^{(1,R)}\right) - \left(\mathbf{x}_0^{(k,L)}\, \mathbf{B}\, \mathbf{x}_0^{(k,R)} - \mathbf{x}_0^{(1,L)}\, \mathbf{B}\, \mathbf{x}_0^{(1,R)}\right) \\ &= \left(\overline{\mathbf{x}}_0^{(k,L)}\, \overline{\mathbf{B}}\, \overline{\mathbf{x}}_0^{(k,R)} - \mathbf{x}_0^{(k,L)}\, \mathbf{B}\, \mathbf{x}_0^{(k,R)}\right) - \left(\overline{\mathbf{x}}_0^{(1,L)}\, \overline{\mathbf{B}}\, \overline{\mathbf{x}}_0^{(1,R)} - \mathbf{x}_0^{(1,L)}\, \mathbf{B}\, \mathbf{x}_0^{(1,R)}\right), \quad k \in \{2, \dots, \overline{S}\}. \end{aligned} \tag{B.31}$$

[8] That is, the multiplicative constant ε is omitted from the right-hand side of (B.31) for notational and exposition simplicities.

We now employ the general identity $\overline{\mathbf{B}} = \overline{\mathbf{A}}_{0\tau}^{-1} \mathbf{A}_{0\tau-} \mathbf{B}^+$ (Proposition 2) and the consistency property (13) of the unperturbed right eigenvectors $\mathbf{x}_0^{(k,R)}$ and $\overline{\mathbf{x}}_0^{(k,R)}$, which share relevant identical components, to reduce $\mathbf{B}\,\mathbf{x}_0^{(k,R)}$ to $\mathbf{B}^+\,\overline{\mathbf{x}}_0^{(k,R)}$, for $k \in \{2, \ldots, \overline{S}\}$, as in (11). As a result, up to the multiplicative constant of ε, the relative spectral gaps of AD price matrices $\overline{\mathbf{A}}$ and $\mathbf{A}$ can be written (similar to (21))

$$RG^{(k)} \equiv \left[\overline{\delta}^{(k)}(\varepsilon) - \overline{\delta}^{(1)}(\varepsilon)\right] - \left[\delta^{(k)}(\varepsilon) - \delta^{(1)}(\varepsilon)\right]$$

$$\sim \underbrace{\left[\overline{\mathbf{x}}_0^{(k,L)}\ \overline{\mathbf{A}}_{0\tau}^{-1}\mathbf{A}_{0\tau-}\ -\ \mathbf{x}_0^{(k,L)}\right]}_{\text{unperturbed \& consistent factor}}\ \underbrace{\mathbf{B}^+\ \overline{\mathbf{x}}_0^{(k,R)}}_{\text{perturbative factor}}\ -\ \underbrace{\left[\overline{\mathbf{x}}_0^{(1,L)}\ \overline{\mathbf{A}}_{0\tau}^{-1}\mathbf{A}_{0\tau-}\ -\ \mathbf{x}_0^{(1,L)}\right]}_{\text{unperturbed \& consistent factor}}\ \underbrace{\mathbf{B}^+\ \overline{\mathbf{x}}_0^{(1,R)}}_{\text{perturbative factor}}, \tag{B.32}$$

for all $k \in \{2, \ldots, \overline{S}\}$. Two observations concerning these relative spectral gaps are in order. While the first shows that both recovery inconsistencies and non-vanishing relative spectral gaps originate from different retentions of information in two inconsistent specifications (Proposition 2), the second shows the difference between recovery inconsistencies and relative spectral gaps.

First observation: In the special case where the perturbative components $\mathbf{B}$ and $\overline{\mathbf{B}}$ of AD price matrices are consistent, Equation (17) implies $\mathbf{A}_{0\tau-}\mathbf{B}^+ = \overline{\mathbf{A}}_{0\tau}\overline{\mathbf{B}}$, and $\mathbf{x}_0^{(k,L)}\mathbf{B}^+ = \overline{\mathbf{x}}_0^{(k,L)}\overline{\mathbf{B}}$, for all $k \in \{1, \ldots, \overline{S}\}$ (as in Footnote 12 of the main text). As a result, both terms on the RHS of (B.32) vanish, reducing to relative spectral gaps $RG^{(k)} = 0$, $\forall k \in \{1, \ldots, \overline{S}\}$ as information about the underlying market model is preserved by consistent perturbative components $\mathbf{B}$ and $\overline{\mathbf{B}}$ (Case 1, Proposition 2). In the general case where the perturbative components of AD price matrices are not consistent with each other, $\mathbf{B}$ and $\overline{\mathbf{B}}$ are not constrained by consistency conditions. Their exogeneity imply that both terms on the RHS of (B.32) do not vanish in general. Furthermore, due to this exogeneity, a finer consolidated specification $\mathcal{S}$ does not unambiguously increase or decrease the correlation between the consistent and inconsistent factors in each of the two terms on the RHS of (B.32). Intuitively, a finer $\mathcal{S}$ does not necessarily reduce the relative spectral gaps $RG^{(k)}$, $k \in \{2, \ldots, \overline{S}\}$, because information about the underlying market model is not preserved by inconsistent perturbative components $\mathbf{B}$ and $\overline{\mathbf{B}}$ (Case 2, Proposition 2).

Second observation: We examine and compare inconsistencies in recovered marginal utilities (22) versus relative spectral gaps of AD price matrices (B.32). Since the unperturbed eigenvalues $\{\delta_0^{(k)}\}$, $k \in \{1, \ldots, S\}$, and $\{\overline{\delta}_0^{(k)}\}$, $k \in \{1, \ldots, \overline{S}\}$, appear explicitly in (22), an important factor to the recovery inconsistencies of marginal utilities is the unpaired terms among the two sums in (22) (indexed by $k \in \{\overline{S}+1, \ldots, S\}$, associated with the original specification). Whereas, since the relative spectral gaps (B.32) concern the comparison of the spectral gaps of AD price matrices, they are about the paired eigenvalues (indexed by $k \in \{1, \ldots, \overline{S}\}$), which differ only in their perturbative components (see (B.30)). As a result, a finer specification affects the recovery inconsistencies in marginal utilities in two channels: (i) the divergence of the paired numerators $\overline{\mathbf{x}}_0^{(k,L)}\,\overline{\mathbf{B}}\,\overline{\mathbf{x}}_0^{(1,R)}$ and $\mathbf{x}_0^{(k,L)}\,\mathbf{B}\,\mathbf{x}_0^{(1,R)}$, $k \in \{1, \ldots, \overline{S}\}$ in (22) due to inconsistent perturbative components of AD price matrices, and (ii) the change in the number of unpaired terms (unpaired contributions) among the two sums in (22). In contrast, a finer specification affects the relative spectral gaps of AD price matrices (B.32) in the first channel as discussed in the first observation above.