Number-phase teleportation and the Heisenberg limit in interferometry: a "paradox" and some surprises

Olivier Pfister* and Ngoc-Khanh Tran

Department of Physics, University of Virginia,
382 McCormick Road, Charlottesville, VA 22904-4714, USA

ABSTRACT

Following previous studies by Milburn and Braunstein, and Cochrane, Milburn, and Munro, we consider number-phase teleportation protocols. We investigate the use, as the teleportation quantum channel, of two-mode states with a perfectly well defined phase difference and number sum, which are also suitable for Heisenberg-limited interferometry. We show that intuition based on squeezing of these variables, which is commonly used to derive entangled states using the EPR paradox, can fail in this case to yield suitable teleportation channels. We show that the domain of failure is in fact of size $1/N$, N being the total number of photons. We also point out another way of generating simpler analogs of number-sum/phase-difference eigenstates.

Keywords: entanglement, quantum information, quantum teleportation, phase-difference eigenstates, EPR states, Heisenberg-limited interferometry

1. INTRODUCTION

It is interesting to investigate quantum information protocols, such as teleportation, using different physical variables. In particular, the use of number and phase variables might be of interest for systems such as Bose-Einstein condensates and trapped ions, for which a number-state description may be more convenient than for photons. In addition, the possible coming of age of on-demand single-photon sources could yield photonic systems that would lend themselves better to a number-state description.

Number-phase teleportation was first studied by Milburn and Braunstein,[1] and by Cochrane, Milburn, and Munro,[2,3] followed by the authors of this paper.[4] In these studies, the quantum teleportation channel is a number-phase EPR state, i.e. a two-mode common eigenstate of the photon number difference $N_- = a^\dagger a - b^\dagger b$ and phase sum $\phi_+ = \phi_a + \phi_b$,[1] or of the photon number sum N_+ and phase difference ϕ_-.[2,4]

In the first case, the (N_-, ϕ_+) EPR state can be taken identical to the two-mode squeezed state that is an (X_-, P_+) EPR state (where $X_a = (a + a^\dagger)/\sqrt{2}$ and $P_a = i(a^\dagger - a)/\sqrt{2}$)

$$|\,\Psi\,\rangle = (\cosh r)^{-1} \sum_{n=0}^{\infty} \tanh^n r \; |\,n\,\rangle_a \otimes |\,n\,\rangle_b \tag{1}$$

and which has been used in recent continuous-variable teleportation experiments.[5–8] It was one of Milburn and Braunstein's points to show that the same state would allow two different teleportation protocols,[1] one using measurements of (X_-, P_+) and one using measurements of (N_-, ϕ_+).

In the second case, the (N_+, ϕ_-) EPR state is[2]

$$|\,\phi = 0\,\rangle = \frac{1}{\sqrt{N+1}} \sum_{n=0}^{N} |\,n\,\rangle_a \otimes |\,N-n\,\rangle_b\,, \tag{2}$$

and is also known as a phase-difference eigenstate.[9] We have shown in Ref. 4 that the most general form of a phase-difference eigenstate, first introduced by Trifonov et al,[10]

$$|\,\Phi\,\rangle = \frac{1}{\sqrt{N+1}} \sum_{n=0}^{N} e^{in\phi_n} \; |\,n\,\rangle_a \otimes |\,N-n\,\rangle_b\,, \tag{3}$$

* Corresponding author. E-mail: opfister@virginia.edu, Telephone: 1 434 924 7956

yields unity teleportation fidelity as well. In Eq. (3), the phase distribution $\Phi = \{\phi_n\}_n$ may be anything, but either Alice or Bob must have complete knowledge of it for the teleportation protocol to be successful.

One is tempted to draw an analogy with the (N_-, ϕ_+) case and consider the EPR quadratures (X_+, P_-), but it is easy to show that the state $|\,\Phi\,\rangle$ is neither an eigenstate of the amplitude-quadrature sum X_+ nor of the phase-quadrature difference P_-. For the state of Eq. (2), the EPR variances are

$$V(X_+) \;=\; N+1 \;+\; 2\sum_{k=0}^{N}\sqrt{\frac{k}{N+1}\left(1-\frac{k}{N+1}\right)} \;\overset{N\to\infty}{\longrightarrow}\; N\left(1+\frac{\pi}{4}\right) \tag{4}$$

$$V(P_-) \;=\; N+1 \;-\; 2\sum_{k=0}^{N}\sqrt{\frac{k}{N+1}\left(1-\frac{k}{N+1}\right)} \;\overset{N\to\infty}{\longrightarrow}\; N\left(1-\frac{\pi}{4}\right). \tag{5}$$

This gives us a "non-EPR" Heisenberg-like inequality

$$V(X_+)V(P_-) \;=\; N^2\left(1-\frac{\pi^2}{16}\right) > 0, \tag{6}$$

and for the Duan entanglement criterion[11]

$$V(X_+) + V(P_-) \;=\; 2N > 2. \tag{7}$$

There is therefore no clear correspondence between number-phase entanglement and quadrature entanglement for the number-sum, phase-difference case, contrary to the opposite case studied in Ref. 1. We do know, however, that the states $|\,\Phi\,\rangle$ are maximally entangled (the partial trace of their density matrix gives the identity matrix) and have perfectly well defined number-sum (obvious) and phase-difference, defined in Ref.9.

$$V(N_+) = V(\phi_-) \;=\; 0. \tag{8}$$

The teleportation protocol for these states is based on measurements of the corresponding variables and has been described in detail in Ref.2, and in Ref.4 for generalized states $|\,\Phi\,\rangle$. Such states have not yet been created in the lab for $N > 2$ (see Ref.10 for $N = 2$). It is therefore of interest to find some ways of finding close equivalents to number-phase eigenstates that could be created experimentally with large N. In the next section, we investigate this problem starting with an "EPR" reasoning based on Heisenberg-limited interferometry, that paradoxically proves unsuccessful.

2. FINDING (N_+, ϕ_-) TELEPORTATION CHANNELS: A PARADOX

2.1. Heisenberg-limited interferometry and the balanced lossless beam splitter

Phase-difference eigenstates are obviously of interest for ultra-sensitive interferometric measurements, whose ultimate limit can be shown to be the Heisenberg limit[12–14]

$$V(\phi_a - \phi_b) \;\sim\; \frac{1}{N}, \tag{9}$$

as opposed to the standard-shot-noise limit

$$V(\phi_a - \phi_b) \;\sim\; \frac{1}{\sqrt{N}}. \tag{10}$$

The field of Heisenberg-limited interferometry (HLI. See, for example, Refs.15, 16 for a review) therefore overlaps with quantum information, as (N_+, ϕ_-) states provide a teleportation channel as well as enable one to reach the HL. It is interesting to see whether knowledge about HLI could help with the problem at hand of finding (N_+, ϕ_-) states. In particular, the physics of the beam splitter suggests a simple method, exposed below.

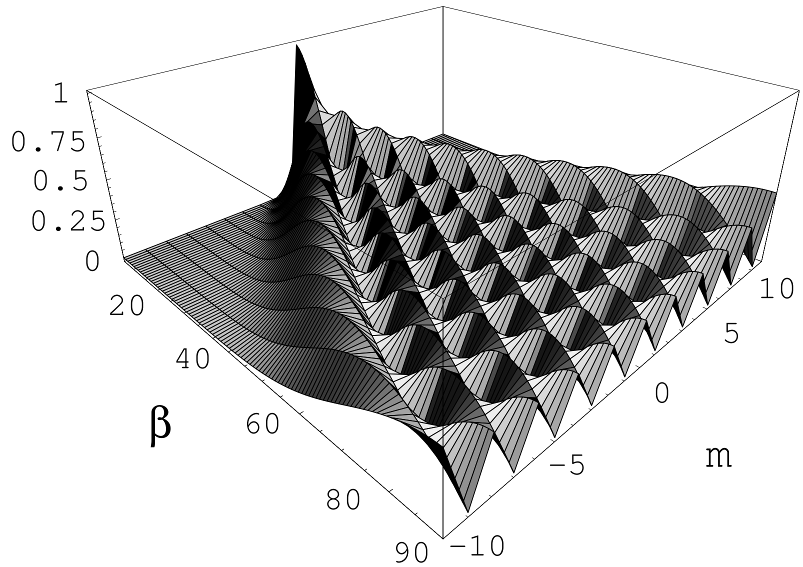

Figure 1. Modulus of the quantum amplitudes of $|\,\psi_{10}(\beta)\,\rangle$, versus beam-splitter angle β in degrees. $\beta = 0$ corresponds to $R = 0$ or 1.

2.2. An entanglement conundrum

Several authors[17–19] have shown, using different methods, that a lossless balanced beam splitter swaps the number- and phase-difference quantum fluctuations between its input and output ports. In other words, if a number-difference squeezed input is used, a phase-difference squeezed output is created. Now, since a lossless beam splitter is unitary, a number-sum eigenstate must stay so going through the beam splitter, because of energy conservation. Therefore, if one started with a state of fixed total photon number and number-difference squeezed, then the beam splitter should transform it into a state of same fixed total photon number and phase-difference squeezed, i.e. an EPR state, i.e. a suitable teleportation resource.

The last phrase of the previous paragraph is not true, as is shown by the simple example of a twin Fock state $|\,n\,\rangle_a|\,n\,\rangle_b$. The action of the beam splitter on this state has first been calculated by Campos, Saleh, and Teich,[20] and in many works since them. A convenient way to calculate it is to use rotation matrices in the Schwinger representation[17,21]

$$\left|\,\psi_n(\frac{\pi}{2})\,\right\rangle \;=\; e^{-\frac{i\pi}{4\hbar}(a^\dagger b + ab^\dagger)}\;|\,n\,\rangle_a|\,n\,\rangle_b \;=\; \sum_{m=-n}^{n} d^n_{m0}(\frac{\pi}{2})\;|\,n+m\,\rangle_a|\,n-m\,\rangle_b\,, \tag{11}$$

which yields the well-known result that the state has about half of its amplitudes nulled by the fact that $d^n_{m0}(\frac{\pi}{2}) = 0$ if m and n have opposite parities. (See Fig. 1 at $\beta = 90^o$.) The state $\left|\,\psi_n(\frac{\pi}{2})\,\right\rangle$ is therefore not maximally entangled,[22] and its maximum teleportation fidelity is 50% for a coherent state.[2] Cochrane, Milburn, and Munro showed, however, that $\left|\,\psi_n(\frac{\pi}{2})\,\right\rangle$ is a good teleportation resource for teleporting cat-like states such as $|\,\alpha\,\rangle + |\,-\alpha\,\rangle$, which contain nonzero amplitudes for a given photon number parity only. Hence, an (N_+, ϕ_-) EPR state like $\left|\,\psi_n(\frac{\pi}{2})\,\right\rangle$ is not a perfect teleportation channel. This is what we call the "paradox" here.

2.3. An explanation

We do not know the reason for this "paradox," but we know how it happens: the nulling of every other quantum amplitude of $\left|\,\psi_n(\frac{\pi}{2})\,\right\rangle$ is a quantum interference effect involving the whole $2n+1$-dimensional Hilbert space and is the multi-photon analog of the well-known quantum interference in the Hong-Ou-Mandel interferometer,[23] by

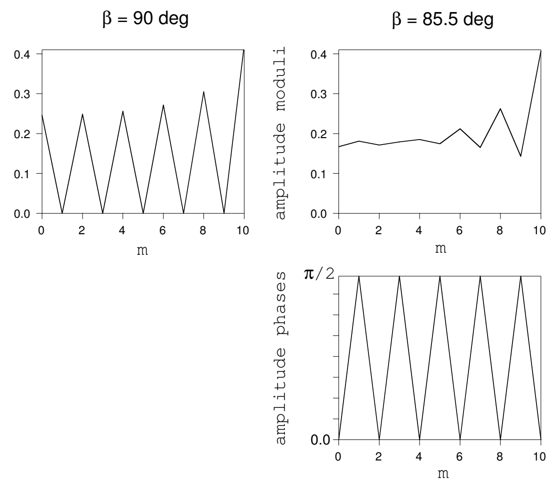

Figure 2. Modulus and phase of the quantum amplitudes of $|\,\psi_{10}(\pi/2)\,\rangle$ and $|\,\psi_{10}(\beta_c)\,\rangle$, versus m. All nonzero amplitudes of $|\,\psi_{10}(90^{o})\,\rangle$ have a phase equal to $\pi/2$.

which two identical photons impinging on different ports of a beam splitter always exit from one same port. This effect is calculable using Eq. (11)

$$\left|\psi_1(\frac{\pi}{2})\right\rangle \;=\; e^{-\frac{i\pi}{2\hbar}(a^\dagger b+ab^\dagger)}\;|\,1\,\rangle_a\,|\,1\,\rangle_b \;=\; \sum_{m=-1}^{1} d^1_{m0}(\frac{\pi}{2})\;|\,1+m\,\rangle_a\,|\,1-m\,\rangle_b \;=\; \frac{1}{\sqrt{2}}\,[|\,0\,\rangle_a\,|\,2\,\rangle_b - |\,2\,\rangle_a\,|\,0\,\rangle_b]. \quad (12)$$

An equivalent explanation is the fact that the two probability amplitudes of output state $|\,1\,\rangle_a\,|\,1\,\rangle_b$ (both photons transmitted or both reflected) interfere destructively when the photons are indistinguishable. This effect, first evidenced with down-converted pairs from a nonlinear crystal,[23] has recently been demonstrated with two single photons coming from the same source.[24]

2.4. A solution

One could conjecture that, for n photons in each port, the destructive quantum interference could behave in a manner analogous to a n-slit diffraction grating, i.e. the interference could become extremely sharp in the beam splitter "angle", defined as $\beta = \pm 2\arccos\sqrt{R}$ (R: reflectivity). If the effect is as sharp as $\Delta\beta \sim 1/n$, it could become extremely hard to observe. To ascertain this, we plot the quantum amplitudes of $|\,\psi\,\rangle_n(\beta)$ versus β, in Fig. 1. For β much smaller than $\pi/2$, the spread of the state decreases, which means it becomes less than maximally entangled. One can see, however, that, close to $\pi/2$, the nulling of the amplitudes disappears very quickly but the spread doesn't decrease. To determine the size of this region in terms of β, we look for the crtitical value β_c closest to $\pi/2$ such that the quantum amplitudes look as constant as possible but the spread of the state is still the $(2n+1)$-D Hilbert space. For $n = 10$, we plot the corresponding cuts for $\beta = \pi/2$ and β_c, on Fig. 2. The state $|\,\psi_{10}(85.5^o)\,\rangle$ looks extremely close to a maximally entangled state, even though its phase distribution is now nontrivial (but this only makes it a generalized phase-difference eigenstate $|\,\Phi\,\rangle$). Following our hypothesis, we postulate that

$$\beta_Q \;=\; \frac{\pi}{2}\left(1-\frac{1}{2n}\right). \quad (13)$$

which yields

n	10	100	1000	10000
β_Q (^{o})	85.5	89.55	89.955	89.9955

The corresponding amplitudes are plotted on Fig. 3, which confirms that the nulling of the amplitudes has disappeared and that the spread of the state is still maximal. This is a numerical validation of our hypothesis:

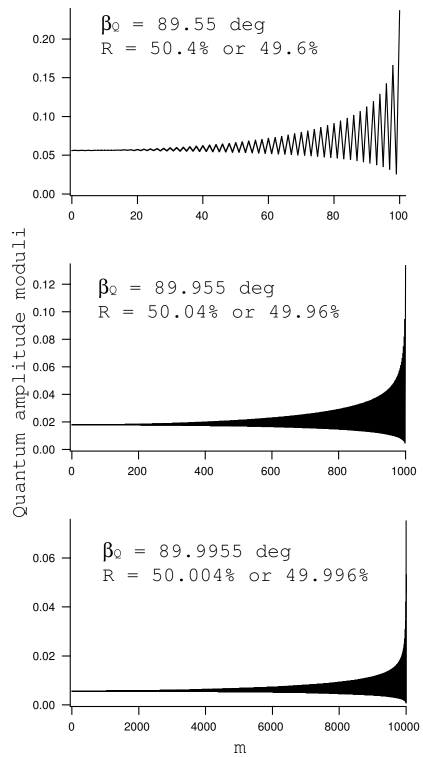

Figure 3. Modulus of the quantum amplitudes of $|\,\psi_n(\beta_c)\,\rangle$, versus m, for $n = 100$, 1000, and 10000.

the nulling of the amplitudes would probably be very hard to obtain experimentally with a large number n of bosons per beam splitter port. Finally, we analyze the teleportation fidelity of entanglement channel $|\,\psi_{10}(\beta)\,\rangle$ in the teleportation of the coherent state $|\,\alpha = 3\,\rangle$. The fidelity is plotted versus β on Fig. 4. One clearly sees a

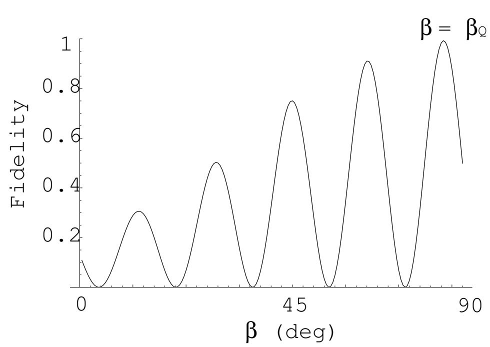

Figure 4. Fidelity of the teleportation of $|\,\alpha = 3\,\rangle$ using the entanglement channel $|\,\psi_{10}(\beta)\,\rangle$. Note the 50% value at $\pi/2$ and the 99.3% value at β_Q.

99.3% maximum at $\beta = \beta_Q$ and the 50% value of Ref.2 at $\beta = \pi/2$.

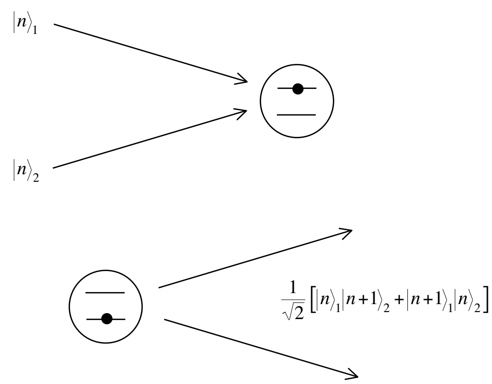

Figure 5. Generation of the state $|\chi\rangle$.

2.5. Another solution

We present here another method to obtain a working approximation of a perfect (N_+, ϕ_-) eigenstate. The idea is simply to send the desired output state (3) into a beam splitter and analyze the result.[4] Doing this, one finds the simple state

$$|\chi\rangle \ = \ \frac{1}{\sqrt{2}}\,[|\,n\,\rangle_a\,|\,n+1\,\rangle_b + |\,n+1\,\rangle_a\,|\,n\,\rangle_b]. \tag{14}$$

Such a state could be generated from a twin Fock pair (outputs of two on-demand single-photon sources) impinging on a single, long-lived, excited atom followed by a balanced beam splitter (Fig.5). One could also think of twin boson Fock states interacting with a single quantum of interaction between the modes before undergoing a $\pi/2$ pulse analogous to a beam splitter: the two boson modes could therefore be two condensates in two different electronic states interacting with a single resonant photon followed by a $\pi/2$ laser pulse.

This state is transformed by a beam splitter into a close analog of the ideal EPR state, as displayed on Fig. 6 The teleportation fidelity for this state is also close to 100%.[4] In addition, the protocol is simpler due to the

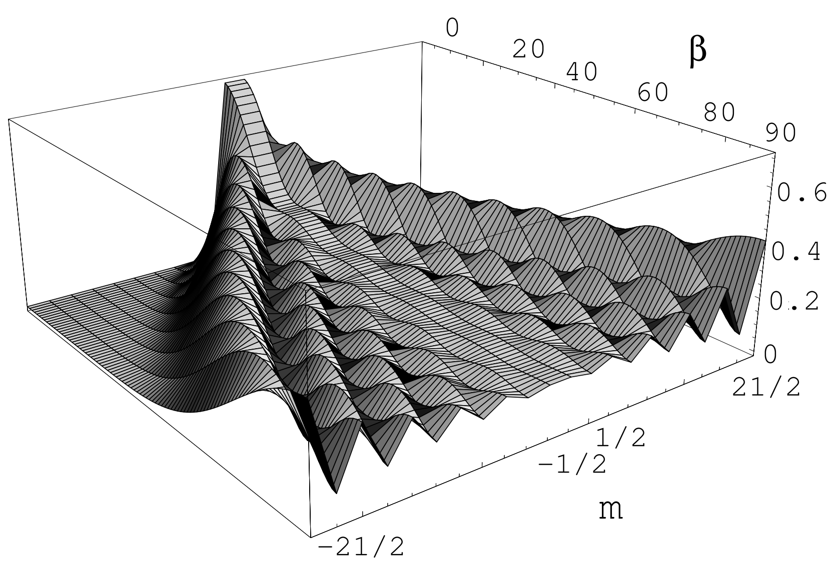

Figure 6. Modulus of the quantum amplitudes of $|\chi(\beta)\rangle$, versus beam-splitter angle β in degrees.

regular phase distribution of the state.

In conclusion, we have studied the creation of number-phase entanglement using a beam splitter. We have shown that quantum interferences can sneak into qualitative reasoning to yield surprising — albeit well known — results. We show for the first time, to our knowledge, that the effect of the interference is visible over a parameter range inversely proportional to the particle number, which could make it extremely difficult to observe, in fact, for large such numbers. Finally, we have shown that it is possible to find good approximations to number-phase EPR states, some of which might even be experimentally realizable.

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