

Affine Disagreement and Asset Pricing

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Models of heterogeneous beliefs can generate rich implications for trading and asset pricing (see Suleyman Basak (2005) for a recent survey). When studying such models, aggregation often leads to difficulty in computing equilibrium outcomes. In this paper, we introduce a flexible framework to model heterogeneous beliefs in the economy, which we refer to as “affine disagreement” about fundamentals. Affine processes (see Darrel Duffie, Jun Pan, and Kenneth Singleton (2000)) are appealing as they provide a large degree of flexibility in modelling the conditional means, volatilities, and jumps for various quantities of interest while remains analytically tractable. Our affine heterogeneous beliefs framework allows further for stochastic disagreement among agents about growth rates, volatility dynamics, as well as the likelihood of jumps and the distribution of jump sizes.

Disagreement about rare disasters provides an interesting case study for our framework. Research by Thomas A. Rietz (1988), Francis Longstaff and Monika Piazzesi (2004), Robert J. Barro (2006) and others show that the possible occurrence of rare disasters that result in severe losses in consumption can have large impact on asset prices. However, the relatively short sample period and lack of historical precedents of disaster events (at least in the US) make it difficult to precisely measure the frequencies of disasters or the size of their impact. Together, these suggest that there is likely to be large disagreements among market participants about disasters, and such disagreements can significantly affect asset prices.

A number of interesting implications arise from heterogeneous beliefs about disasters. The model

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endogenously generates variation in the risk-free rates, asset prices, and the equity risk premium through variation in the distribution of wealth. In normal times, optimistic agents (who believe disasters are less frequent and likely to be less severe) accumulate wealth, which leads to a gradual decline in the equity premium. When disasters strike, the pessimistic agents become relatively more wealthy, resulting in jumps in the equity premium.

I. An Affine Heterogeneous Beliefs Framework

We consider an endowment economy. The stochastic environment is summarized by the Markov state variable X_t , which reflects information about both the aggregate endowment and agents’ beliefs. We now show how one can choose X_t to model a broad class of disagreement over the dynamics of the economy.

A. Beliefs

There are two agents (A, B), each being the representative of her own class, who possess heterogeneous beliefs about the dynamics of X_t . Agent A believes that X_t follows an affine jump diffusion:

$$dX_t = \mu_t^A dt + \sigma_t^A dW_t^A + dJ_t^A,$$

where W_t^A is a standard Brownian motion, $\mu_t^A = K_0^A + K_1^A X_t$, and $\sigma_t^A (\sigma_t^A)^\top = H_0^A + H_1^A \cdot X_t$. The term J_t^A is a pure jump process with intensity $\lambda_t^A = \lambda_0^A + \lambda_1^A \cdot X_t$, and its jump size has distribution ν^A , with moment generating function ϕ . We summarize agent A’s beliefs with the probability measure \mathbb{P}_A . For simplicity, we suppose that A’s beliefs are correct. The method is easily modified to the case where neither agent has correct beliefs.

Agent B has an equivalent probability measure \mathbb{P}_B . The differences in beliefs are characterized by the Radon-Nikodym derivative $\eta_t \equiv E_t[d\mathbb{P}_B/d\mathbb{P}_A]$.

We assume that

$$(1) \quad \eta_t = e^{a \cdot X_t - I_t},$$

where I_t is locally deterministic satisfying

$$(2) \quad \frac{dI}{dt} = a \cdot \mu_t^A + \frac{1}{2} \|\sigma_t^A a\|^2 + \lambda_t^A (\phi(a) - 1),$$

which ensures that η_t is a \mathbb{P}_A -martingale.

It follows from the specification of η_t that, under B's beliefs, X_t follows an affine jump diffusion

$$dX_t = \mu_t^B dt + \sigma_t^B dW_t^B + dJ_t^B,$$

where W_t^B is a standard Brownian motion under \mathbb{P}_B , and

- i. $\mu_t^B = \mu_t^A + \sigma_t^A (\sigma_t^A)^\top a$
- ii. $\sigma_t^B = \sigma_t^A$
- iii. $\lambda_t^B = \phi(a) \lambda_t^A$
- iv. $d\nu^B / d\nu^A(z) = e^{az} / \phi(a)$

Intuitively, the Radon-Nikodym derivative expresses the differences in beliefs by having agent B assign a higher (lower) probability than A to those states where η_t is high (low). For example, if η_t is increasing with a component of X_t , then B thinks that higher values of this component are more likely than A. In other words, B believes the drift for this component is larger. Similarly, if η_t jumps at the same time when X_t jumps, then A and B will disagree about the likelihood of jumps. In particular, if the jump in η_t is positive, then B believes that the likelihood of such a jump is higher than A. Moreover, if the jump size in η_t varies with the jump size in X_t , then A and B will disagree about the jump size distribution as well. Thus, this setup can accommodate both disagreement about the frequency of jumps as well as the conditional distribution of jump sizes.

Finally, while we specify the differences in beliefs exogenously (agents "agree to disagree"), this does not preclude agents' beliefs from arising through Bayesian updating based on different information sets. For example, when the state variables and signals follow a Gaussian process, Bayesian updating can lead to heterogeneous beliefs in the form of (1-2).

B. Equilibrium Asset Prices

We assume that the agents have constant relative risk aversion (CRRA) preferences:

$$U^i(C) = E_0^{\mathbb{P}^i} \left[\int_0^\infty e^{-\rho t} (C_t^i)^{1-\gamma} / (1-\gamma) dt \right],$$

for $i = A, B$. In addition, we assume that (i) markets are complete, (ii) log aggregate consumption, $c_t = \log(C_t)$, is linear in X_t ($c_t = c \cdot X_t$), and (iii) agents are endowed with some fixed fraction ($\theta_A, \theta_B = 1 - \theta_A$) of aggregate consumption.

We first solve for the equilibrium consumption allocations through the planner's problem, and then use the individual consumption for agent A to determine the stochastic discount factor with respect to her beliefs. The equilibrium consumption of agent A is

$$C_t^A = \frac{1}{1 + \tilde{\zeta}_t^{1/\gamma}} C_t,$$

where $\tilde{\zeta}_t = \zeta \eta_t$ is the stochastic weight that the planner places on agent B, which is linked to the initial allocation of wealth and the differences in beliefs. The stochastic discount factor (SDF) under A's beliefs is

$$(3) \quad M_t = e^{-\rho t} (C_t^A)^{-\gamma} = e^{-\rho t} \left(1 + \tilde{\zeta}_t^{1/\gamma} \right)^\gamma C_t^{-\gamma}.$$

With this stochastic discount factor, we can price a large class of assets (e.g. riskless bonds and the aggregate consumption claim) either directly or through the generalized transform in Chen and Joslin (2009), which gives closed form solutions to nonlinear moments of affine processes, such as the SDF in (3). Additionally, the stochastic discount factor characterizes both the instantaneous short rate and the risk neutral measure (\mathbb{Q}). Computation of the risk neutral dynamics is convenient since it allows one to compute the risk premium for any asset (expressed as a difference in conditional means under the \mathbb{P} and \mathbb{Q} measures.)

II. Examples of Heterogeneous Beliefs

The framework above can accommodate a wide range of specifications of heterogeneous beliefs regarding fundamentals such as growth rates or

volatility, likelihood of jumps, or distribution of jumps. It can also be used to model divergence in higher order beliefs. We now discuss our main example of disagreement about disasters as well as a few further examples within the affine framework.

A. Disagreement about Disasters

Suppose agent A believes that the aggregate endowment is $C_t = e^{c_t^c + c_t^d}$, where c_t^c is the diffusion component of log aggregate endowment,

$$dc_t^c = \bar{g}dt + \sigma_c dB_t^c,$$

and c_t^d is a pure jump process whose jumps arrive with constant intensity λ^A . The jump size has distribution ν^A . These jumps in c_t^d capture the impact of disasters on aggregate consumption.

Agent B can disagree about the intensity (λ^B) and/or the severity (ν^B) of disasters. Her beliefs are characterized by the Radon-Nikodym derivative,

$$\eta_t = e^{a_t - (\lambda^B - \lambda^A)t},$$

where a_t is a pure jump process whose jumps are coincident with the jumps in c_t^d and have size

$$\Delta a_t = \log \left(\frac{\lambda^B}{\lambda^A} \frac{d\nu^B}{d\nu^A} \right),$$

which is the log likelihood ratio for a disaster of a given size under the two agents' beliefs. Thus, if B believes disasters of a certain type are more likely than A (either due to her beliefs on the intensity or the jump size distribution), whenever such a disaster occurs, the Radon-Nikodym derivative jumps up, which makes B assign a higher probability to such an event.

This setting remains within the affine family as the state variable $X_t = (c_t^c, c_t^d, \log \eta_t)$ follows a jointly affine process, where the moment generating function of the jumps in X_t can be computed using the moment generating function of the disasters.

B. Other Examples

Besides disaster risks, we can also use the affine heterogeneous beliefs framework to model disagreements about a variety of conditional moments of

the aggregate endowment and other random variables in the economy.

- 1) Disagreement about the growth rates of consumption. Such disagreement in growth rates can be stochastic. For example, Bernard Dumas, Alexander Kurshev, and Raman Uppal (2009) assume that the disagreement about growth rate itself follows a mean-reverting process. Chen and Joslin (2009) explain how to map this model (as well as more general models of disagreements about growth rates) into the affine framework.
- 2) When the endowment process has stochastic volatility, agents can disagree about the dynamics of future volatility. Similarly, they can disagree about future growth rates, or future jump intensities.
- 3) Another source of disagreement is the prospects for future disagreement. Disagreements about higher order beliefs can be captured by our framework since the beliefs are represented within the state variable X_t .

III. A Calibrated Example

In this section, we calibrate a model with heterogeneous beliefs about rare disasters, which is a special case of Chen, Joslin, and Ngoc-Khanh Tran (2010). We assume that the two agents not only disagree about the frequency of disasters, but also the conditional distribution of disaster size. Specifically, agent A believes that disasters have constant intensity $\lambda^A = 0.025$ (on average once every 40 years), and the disaster size follows a binomial distribution, where log consumption drop can be $d_1 = -0.3$ or $d_2 = -0.55$ with equal probability ($p_1^A = p_2^A = 0.5$), conditional on a disaster occurring. Agent B is more optimistic about disasters, in that she believes the intensity is lower, $\lambda^B = 0.02$, and that small disasters are more likely ($p_1^B = 0.75, p_2^B = 0.25$). They agree on the expected growth rate and volatility of consumption without a disaster, $\bar{g} = 0.025$, $\sigma_c = 0.02$, and have the same preferences, $\rho = 0.02$, $\gamma = 4$.

The equity premium is different under the two agents' beliefs. Throughout this section, we report the premium under A's beliefs. In an economy fully occupied by agent A, the risk premium

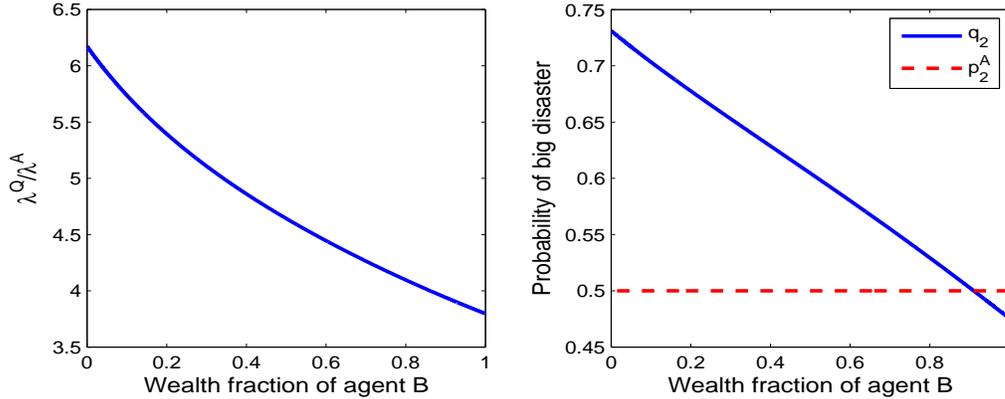


FIGURE 1. **Heterogeneous beliefs, jump-risk premium, and the risk-neutral jump distribution.**

for the aggregate endowment claim is 5.2 percent. Since agent B is more optimistic, when she has all the wealth, the risk premium falls to 2.5 percent. Besides the standard premium for bearing brownian risk (which is small), there is also compensation for jump risks, which accrues due to the covariation between the marginal utility of individual consumption and return on the consumption claim in a disaster.

When both agents are present in the economy, their different beliefs about disasters generate trading and risk sharing. The ability for an agent to trade is limited by her wealth. Thus, the equilibrium disaster risk exposure of each agent and the premium they demand also depend on the wealth allocation between the two agents.

With the majority of the equity premium in this model due to the risk of disasters, it is informative to examine the disaster intensity and the distribution of disaster size under the risk-neutral probability. The left panel of Figure 1 plots the jump-risk premium for agent A, measured by the ratio of the risk-neutral disaster intensity λ^Q (same for both agents) and the disaster intensity under agent A's beliefs λ^A . When A has all the wealth, the likelihood of a disaster under the risk-neutral probability is over 6 times as high as the actual likelihood she believes in, which also indicates the degree to which she values insurance against disasters. This jump-risk premium falls gradually as agent B gains relatively more wealth.

We can also decompose the jump-risk premium for disasters of a given size,

$$\frac{\lambda_t^Q q_{j,t}}{\lambda^A p_j^A} = e^{-\gamma \Delta c_{j,t}^A}, \quad j = 1, 2$$

where $q_{j,t}$ is the risk neutral conditional probability of disaster size d_j , and $\Delta c_{j,t}^A$ is the jump size of the equilibrium consumption for agent A at a time when a disaster of size d_j strikes. Because of trading, the jumps in individual consumption can be very different from the jumps in aggregate endowment. The right panel of Figure 1 compares the risk-neutral probability of a big disaster (conditional on a disaster occurring) with the actual probability under A's beliefs. While A believes that the big disaster is as likely as the small one ($p_2^A = 0.5$), under risk-neutral probability, she distorts the probability of the big disaster as a way to adjust for the risks. For example, when A has all the wealth, the risk-neutral conditional probability of a big disaster is $q_2 = 0.73$. Trading with agent B helps reduce the consumption loss for A during a big disaster, thus lowering the risk-neutral probability q_2 . When B has most of the wealth, A can reduce her consumption loss in a big disaster to the extent that q_2 becomes smaller than p_2^A .

The risk-neutral jump intensity of a specific disaster $\lambda_t^Q q_{j,t}$ has another interesting interpretation. It is approximately the premium of a 1-year disaster insurance, which pays \$1 at the time when a

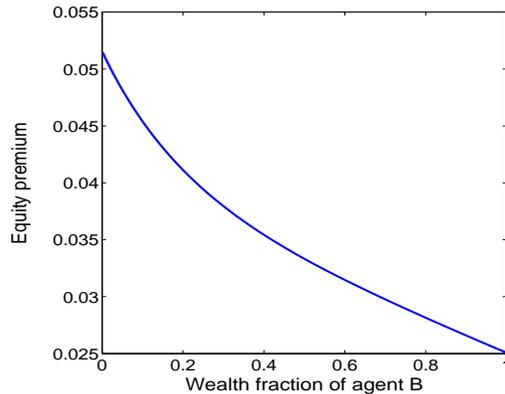


FIGURE 2. Heterogeneous beliefs and the equity premium under agent A’s beliefs.

disaster of size d_j occurs. For example, when agent A has all the wealth, the annual premium for \$1 protection against the big disaster is 11 cents, even though the chance of payoff is only 1.25%. When wealth is evenly distributed between the two agents, the insurance premium falls to 7 cents.

Figure 2 shows the conditional equity premium under agent A’s beliefs as the wealth allocation in the economy changes. At the left and right boundaries, the equity premium converges to the values in the corresponding single-agent economy. As the optimistic agent (B) becomes relatively wealthier, the equity premium falls. When the fraction of total wealth agent B owns rises from 0 to 50%, the equity premium falls from 5.2% to 3.4%.

The time-variation in the equity premium is endogenous. It results from changes in the wealth allocation between the two agents, as opposed to the exogenous variation in the disaster intensity as in Xavier Gabaix (2009) or Jessica Wachter (2009). Due to her more optimistic beliefs, agent B’s wealth will gradually rise relative to agent A at times when a disaster does not occur, which tends to drive the equity premium in the economy lower. When a big (small) disaster occurs, B will lose (gain) wealth relatively, and the equity premium will rise (fall).

IV. Conclusion

We present an affine heterogeneous beliefs framework, where agents may disagree about the growth

rates, dynamics of volatility, jump intensities, or jump size distributions of fundamentals. The flexibility of this tractable framework allows us to study various types of disagreements and their impacts on asset prices. One example we consider is when agents disagree about the frequency of disasters as well as the distribution of consumption losses in a given disaster. The model generates endogenous time variation in the equity premium, linking it to the wealth distribution among agents with different beliefs.

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